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Compressibility Effects in Aerodynamics

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Introduction

A few decades ago the most enthusiastic admirer of mathematical analysis would not have expected that practical engineers engaged in the design of aircraft would have so much use for the mathematical theory of fluid motion as is the case in modern aeronautical engineering. The various applications of the theory are based almost entirely on the hydrodynamics of incompressible perfect fluids, a discipline that half a century ago was considered as an interesting field of pure science having very little to do with the motion which actually takes place in a real fluid. Recently, interest has been centered on another branch of fluid mechanics, namely, on the mechanics of compressible fluids. The aeronautical engineer is pounding hard on the closed door leading into the field of supersonic motion. He realizes that the price that has to be paid for further increase of speed becomes higher and higher if he nears this frontier. However, he wonders whether the mathematical theory could not be used as a guide for avoiding a premature drop of aerodynamic efficiency. The present paper has the objective of reviewing the present status of the theory of compressible fluids from the practical standpoint of its usefulness for interpretation of experimental research and guidance in design.

Irrotational Motion of Ideal Fluids

The problem of irrotational motion of ideal incompressible fluids is more or less a problem of pure geometry. Every velocity field that satisfies the conditions of no vorticity (zero rotation) and the condition of continuity (zero divergence) represents a possible, *i.e.*, dynamically correct, flow pattern of an ideal incompressible fluid. For example, in the case of two-dimensional motion, any network of two families of orthogonal curves represents a network of streamlines and equipotential lines of a dynamically correct flow pattern provided it satisfies one condition: that all its elements are of square shape. The flow is completely determined by the rule that the amount of fluid passing between two neighboring streamlines is constant and the "potential difference" between two successive equipotential lines is a constant quantity. If

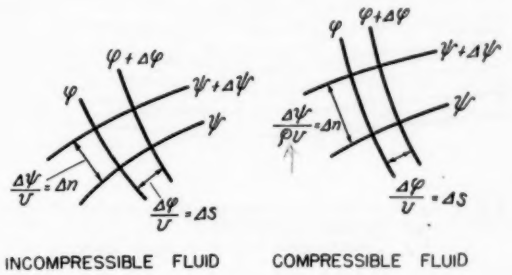

Fig. 1.

the fluid volume passing through an elementary channel of width Δn (Fig. 1) is denoted by $\Delta\psi$, obviously $\Delta\psi = v\Delta n$, and if the potential difference between two successive cross-sections at the distance Δs is denoted by $\Delta\varphi$, obviously $\Delta\varphi = v\Delta s$. If one chooses for convenience $\Delta\varphi = \Delta\psi$, it is seen that $\Delta n = \Delta s$; *i.e.*, every element $\Delta n\,\Delta s$ of the network has a square shape. If the velocity increases or decreases along a streamline, the elements of the network shrink or expand in such a way that both Δs and Δn are inversely proportional to the velocity. Hence the elements of the network always keep their square shapes.

The pressure does not enter the problem directly, because in the case of incompressibility the pressure is not related to the physical state of the fluid. This can be illustrated, for example, by the fact that provided the fluid has no free surface (*i.e.*, a surface of given constant pressure) one can superpose or remove a gravity

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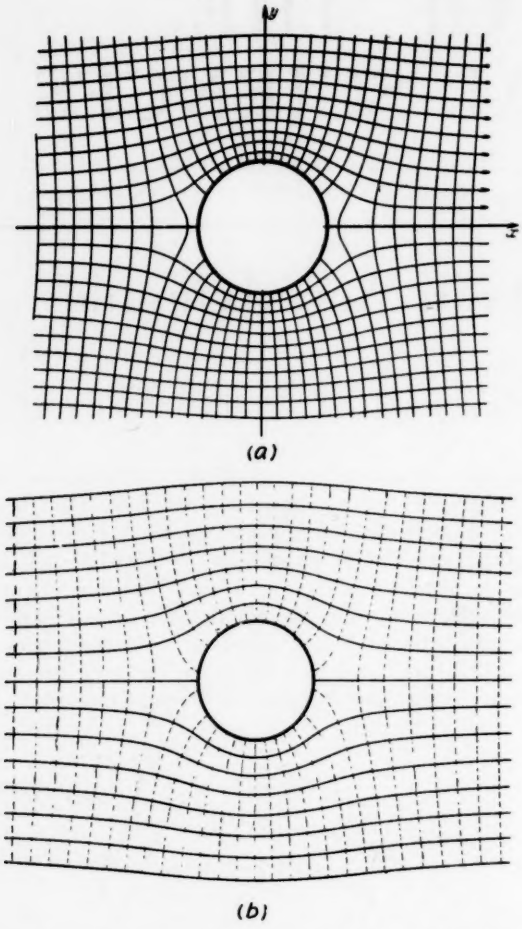

FIG. 2. φ, ψ—networks for flow around a circular cylinder.
(a) Incompressible fluid.
(b) Compressible fluid with $M_1 = 0.400$.

field on the fluid in motion without changing the flow pattern or the velocity distribution. The gravity field produces a pressure field which is simply superposed upon the pressures that keep the fluid in dynamic equilibrium. This is correct not only in the case of gravity but also in the case of any field of conservative forces.

In dealing with compressible fluids two new aspects have to be taken into account:

(a) The continuity condition for the channels between neighboring streamlines involves the variable density; namely, one has to write:

$$\Delta\psi = \rho v \Delta n \tag{1}$$

where $\Delta\psi$ is now the fluid mass passing through the channel (Fig. 1). The potential difference is equal to $\Delta\varphi = v\Delta s$ as in the case of incompressible fluid, and therefore in a network drawn with a constant ratio between $\Delta\varphi$ and $\Delta\psi$ the ratio $\Delta s/\Delta n$ is variable and proportional to the density ρ. Consequently the elements of the network are elongated in the flow direction if the density increases, and expanded normal to the flow direction if the density decreases. This is clearly seen in Fig. 2 where the φ, ψ networks for the flow of an incompressible and a compressible fluid around a circular cylinder are plotted.

(b) The flow problem cannot be solved by means of geometrical considerations alone. The density is connected with the pressure and temperature by the equation of state. If heat conduction is neglected and it is assumed that all particles have started from the same state at rest, the density and pressure are connected by the "isentropic relation"

$$p = \text{const. } \rho^\gamma \tag{2}$$

where γ is the ratio of the two specific heats (for constant pressure and constant volume) of the fluid. This relation, combined with Bernoulli's equation

$$vdv + (1/\rho)dp = 0 \tag{3}$$

furnishes a relation between density and velocity. This relation has to be introduced into the geometrical conditions of irrotationality and continuity in order to solve the flow problem.

THERMODYNAMIC RELATIONS

If p_0, ρ_0 denote the pressure and density of the fluid at rest, Eq. (2) becomes

$$p = (p_0/\rho_0^\gamma)\rho^\gamma \tag{4}$$

On the other hand, Eq. (3) can be written in the form

$$vdv + (1/\rho)(dp/d\rho)d\rho = 0 \tag{5}$$

The quantity $dp/d\rho$ is equal to the square of the velocity of sound; *i.e.*, the velocity of propagation of infinitesimally small pressure changes. With the notations $dp/d\rho = a^2$ and $(dp/d\rho)_0 = \gamma(p_0/\rho_0) = a_0^2$ (the velocity of sound at rest) the following relations are easily obtained:

$$a^2 = a_0^2 - [(\gamma - 1)/2]v^2 \tag{6}$$

$$\rho = \rho_0\left[1 - \frac{\gamma - 1}{2}\frac{v^2}{a_0^2}\right]^{\frac{1}{\gamma-1}} = \rho_0\left[1 + \frac{\gamma - 1}{2}\frac{v^2}{a^2}\right]^{-\frac{1}{\gamma-1}} \tag{7}$$

$$p = p_0\left[1 - \frac{\gamma - 1}{2}\frac{v^2}{a_0^2}\right]^{\frac{\gamma}{\gamma-1}} = p_0\left[1 + \frac{\gamma - 1}{2}\frac{v^2}{a^2}\right]^{-\frac{\gamma}{\gamma-1}} \tag{8}$$

$$\frac{v^2}{2} = \frac{a_0^2}{\gamma - 1}\left[1 - \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\right] = \frac{a^2}{\gamma - 1}\left[\left(\frac{\rho_0}{\rho}\right)^{\gamma-1} - 1\right] \tag{9}$$

$$\frac{v^2}{2} = \frac{a_0^2}{\gamma - 1}\left[1 - \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right] = \frac{a^2}{\gamma - 1}\left[\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] \tag{10}$$

These relations are well known and are stated here only for the reader's convenience. From (6) and (10) it follows that the so-called critical velocity, *i.e.*, the value of v which is equal to the velocity of sound corresponding to the same pressure and density, and the pressure corresponding to this velocity are given by

$$v_{cr} = \sqrt{2/(\gamma + 1)}\, a_0 \quad p_{cr} = 2/(\gamma + 1)^{\gamma/\gamma-1}\, p_0 \quad (11)$$

For air $\gamma = 1.405$; therefore $v_{cr} = 0.913\ a_0$ and $p_{cr} = 0.527\ p_0$.

For moderate values of v/a_0 the relations (8) can be approximated by the formulas

$$p = p_0 - (\rho_0 v^2/2)\,[1 - (v^2/4a_0^2)]$$

and

$$p = p_0 - (\rho v^2/2)[1 + (v^2/4a^2)]$$

The last equation can be used, for example, to estimate the error made by using for the dynamic pressure $p_0 - p$ the value $\rho v^2/2$ as in the case of incompressible fluids.

Mechanical and Electrical Analogies

(a) Consider the flow of a shallow layer of a heavy, ideal and incompressible fluid. The fluid is bounded by a horizontal plane bottom surface, by vertical walls, and by an upper free surface of constant pressure. The flow of water in a broad channel or in a river is a good example of such a flow, provided the frictional resistance is neglected or it is assumed that the bottom is slightly inclined and the component of gravity in the direction of the flow compensates for the frictional resistance. (This latter assumption can only be realized as far as the average resistance is concerned.) If the variable depth of the fluid layer is denoted by h and the velocity averaged over a vertical line by v, the continuity equation for stationary flow can be written in the form

$$hv\,\Delta n = \Delta\psi = \text{const.} \quad (12)$$

where Δn is the distance between neighboring streamlines. On the other hand, Bernoulli's equation requires that

$$(v^2/2g) + h = \text{const.}$$

Therefore

$$vdv + gdh = 0 \quad (13)$$

Denote the quantity $gh^2/2$ by P; then Eq. (13) can be written in the form

$$vdv + (1/h)dP = vdv + (1/h)(dP/dh)dh = 0 \quad (14)$$

It is seen that the two-dimensional flow of a layer of a compressible fluid of constant depth can be compared with the flow of a layer of variable depth of an incompressible fluid. The Eqs. (12) and (14) are identical with the Eqs. (1) and (3) of a compressible fluid if the depth h is replaced by the density ρ and the quantity P by the pressure p. As a matter of fact, P is equal to the resultant of the pressure forces acting over a section of unit width and depth h for an incompressible fluid of unit density. Furthermore, $dP/dh = gh$ is equal to the square of the wave velocity of long waves propagating in a shallow channel of depth h. This wave velocity takes the place of the velocity of sound in the gas. The so-called hydraulic jump corresponds to the shock wave.

However, the relation between pressure and density of air is different from the relation between P and h. For isentropic change of state, $p = \text{const.}\ \rho^\gamma$, whereas $P = \text{const.}\ h^2$. It is seen that the analogy would be perfect for $\gamma = 2$; however, for air $\gamma = 1.40$ and the highest value of γ (for a monatomic gas) is 1.66. Nevertheless, the analogy is interesting for qualitative information on possible flow patterns, especially in the case of supersonic flow of a gas, which corresponds to the so-called supercritical flow in a channel or river ("torrent"). Many interesting applications have been made by several authors (cf. bibliography at the end of this paper). It seems that E. Jouguet was the first to point out the analogy.

(b) It is known that two-dimensional flow problems for incompressible fluids can be solved by use of a so-called "electrolytic bath" consisting of a layer of liquid of appropriate electrolytic conductivity. Assume that an alternating current passes through the liquid, and denote the amplitude of the alternating voltage by V, the intensity of the alternating current by I, and the conductivity of the liquid by σ. Then the relation between I and the derivative of the voltage in the direction of the current vector is given by

$$I = \sigma(\partial V/\partial s) \quad (15)$$

On the other hand, from the continuity of the electric flux it follows that

$$Ih\,\Delta n = \text{const.} \quad (16)$$

where h is the depth of the liquid layer and Δn is the normal distance between neighboring lines drawn orthogonal to the lines of constant voltage.

It is seen that the Eqs. (15) and (16) are analogous to the fundamental equations for irrotational flow of an incompressible fluid if V/σ is replaced by the velocity potential and I by the velocity v. If h is variable, Eq. (16) is identical with the continuity equation for a gas whose density at an arbitrary point is proportional to the value of h at the same point. Hence the "electrolytic bath" can be used for solution of problems of compressible flow by means of an iteration method. One can first solve the problem for $h = \text{const.}$ by measuring, for example, the V distribution for certain given boundary conditions; then ρ can be calculated for the entire field as function of v, and h modified in such a way that $h = \text{const.}\ \rho$. By carrying out the experiment for the modified shape of the setup one obtains a corrected distribution of V and the procedure can be repeated again.

G. I. Taylor and C. F. Sharman have carried out such experiments; the bottom for the liquid layer was made from an insulating material that could easily be carved out to an arbitrary shape. Of course, if the gradient of h is large, the flow of electricity is no longer two-dimensional, and therefore the approximation furnished by the electric experiment is not very good. This occurs especially if the velocity approaches the velocity of sound.

One-Dimensional Gasdynamics

Certain flow problems of compressible fluids can be treated as one-dimensional problems. For example, the flow of a gas through a duct with slightly variable cross-section area can be computed with good approximation by introducing mean values for velocity, pressure and density, the average values being taken over cross-sections of the duct. This method of treatment is similar to the method generally used in hydraulics. Therefore, the theory presented in this section can be called "gasdynamics in hydraulic fashion." Problems in aerodynamics usually do not permit the use of such simplified methods; nevertheless, the one-dimensional case is of interest to the aerodynamicist, since it gives an opportunity to recognize the essential differences between the mechanics of incompressible and compressible fluids.

If v denotes the mean velocity and ρ the mean density of the fluid in an arbitrary cross-section of area S, the flux of mass passing through this cross-section is equal to $S\rho v$. The conservation of matter or the continuity of flow requires that in the stationary case this quantity be constant for all cross-sections and that in the non-stationary case

$$S(\partial\rho/\partial t) + (\partial/\partial x)(S\rho v) = 0 \qquad (17)$$

The coordinate x is taken along the axis of the duct. In the case of stationary flow the velocity and density are connected by the relation (5) (Bernoulli's equation). From the continuity equation $S\rho v$ = const. it follows that

$$(dS/S) + (d\rho/\rho) + (dv/v) = 0 \qquad (18)$$

Substituting from (5) the value of $(d\rho/\rho) = -(vdv/a^2)$, one obtains

$$(d\rho/\rho) + (dv/v) = [1 - (v^2/a^2)](dv/v) \qquad (19)$$

The ratio v/a is called Mach's number M. Then Eq. (18) becomes

$$dS/S = -[(d\rho/\rho) + (dv/v)] = -(1 - M^2)(dv/v) \qquad (20)$$

It is seen that if $M < 1$ the cross-section area of the duct decreases when the velocity increases as in the case of an incompressible fluid; whereas for $M > 1$ the cross-section area increases with increasing velocity. The condition $M = 1$ involves $dS = 0$. For example, in a converging-diverging nozzle the velocity of sound can occur only in the throat section.

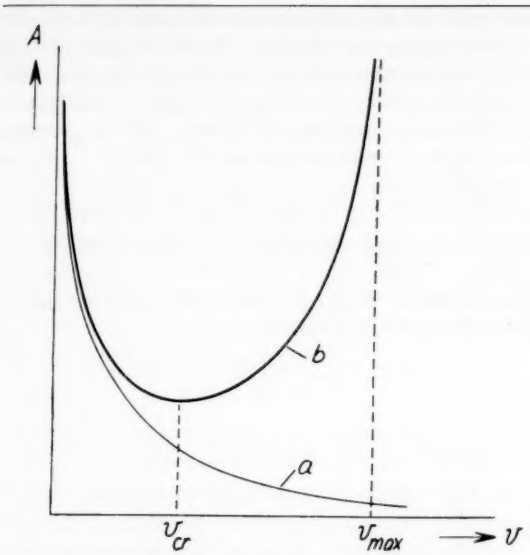

Fig. 3. Relation between the cross-sectional area A and the flow velocity in a nozzle.
(a) Incompressible fluid.
(b) Compressible fluid.

Fig. 3 shows the distribution of cross-section area as function of the velocity v, first for an incompressible fluid (line a), then for a compressible fluid (line b) according to Eq. (20). Assume that a certain nozzle (Fig. 4) is designed for a pressure ratio (p_0/p_e), where p_e is smaller than the critical value p_{cr} given by Eq. (11). In this case the local velocity of sound is reached in the throat cross-section. Then there are two possibilities of flow with isentropic change of state. The gas might expand to the pressure p_e, flowing with supersonic velocity, or it might be compressed isentropically. In the second case the flow is subsonic and the pressure reaches the value p_e' at the exit. The velocity and pressure at the exit are the same as in the cross-section of the same area in the converging portion of the nozzle. The mass flow of gas through the nozzle in the two cases is the same. If the exit pressure is greater than p_e', *i.e.*, it is between p_0 and p_e', the amount of flow varies with the pressure difference between entrance and exit. If the pressure at the exit varies between p_e and p_e', the capacity is constant.

However, the question arises: What happens with the gas if the pressure is varied between these limits? Observation shows that in this case a more or less sudden change of pressure and velocity occurs somewhere. Assume that this shock occurs in a certain cross-section of the nozzle and that the change of state downstream of the shock is isentropic again; then the gas must have a higher value of the entropy than in front of the shock. The possible pressure distribution for constant values of the entropy which are consistent with the given cross-section areas and the given capacity can easily be plotted (Fig. 4). The upper branches of such pressure curves correspond to flows with subsonic, the lower

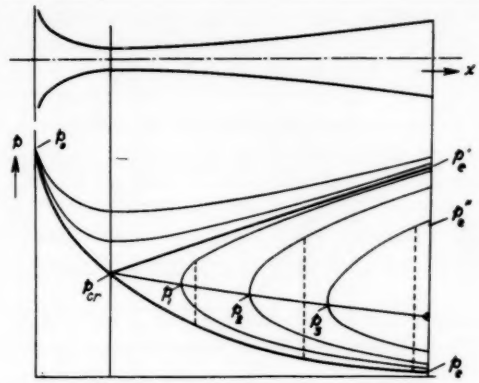

FIG. 4. Pressure distribution along a Laval nozzle.

branches to flows with supersonic velocity. It can be shown that in the case of such a shock the conditions for the conservation of matter, momentum and energy can be satisfied only if the transition takes place between supersonic and subsonic flow. Hence, a jump is physically possible only between points on the initial curve and points on the upper branch of one of the other curves. The location of the shock is determined by the three conditions mentioned. If the end pressure is slightly lower than p_e', the shock occurs close to the throat; if the pressure is lowered, the shock moves toward the exit. There is a value of the end pressure p_e'' that produces a shock right at the exit. If the exit pressure is smaller than p_e'' but greater than p_e, the shock occurs in the free jet in the form of an "oblique" shock wave.

It must be noted that so far the fluid has been considered compressible but non-viscous. Therefore, the results certainly will be modified by viscosity and especially by boundary layer effects. When the nozzle acts as diffusor, *i.e.*, when the flow occurs in the direction of adverse pressure gradient, separation may occur and change the character of the flow and the pressure distribution. However, even if such effects are neglected for the time being, the following conclusions are of interest in view of the general multi-dimensional flow problems discussed in the following sections. In the case of an incompressible fluid, if the pressure at the entrance and exit of a nozzle is arbitrarily given, there always exists a flow with a continuous pressure and velocity distribution which satisfies the conditions of continuity and dynamic equilibrium, the amount of flow being a function of the given pressure difference. In the case of a compressible fluid this is only true if the pressure difference is under a certain limit. Beyond a certain limiting value of the pressure difference the velocity in the nozzle exceeds the velocity of sound and a continuous solution exists only for one definite value of the pressure difference. It appears that in the case of flows that are partially subsonic, partially supersonic, the existence of continuous velocity fields is one of the questions of basic importance.

STATIONARY FLOW AROUND A BODY

The main problem of primary interest for aeronautical applications is that of the stationary flow around a body. If the body is approximately of cylindrical shape, like an airplane wing, the flow can be considered as two-dimensional. The flow around a nacelle, fuselage or cowling offers examples of typical three-dimensional problems. In some cases the assumption of axial symmetry simplifies the problem.

A multi-dimensional flow can be considered as consisting of elementary channels of infinitesimal cross-section area. For each of these channels the continuity equation and Bernoulli's equation must hold. However, since the channels are not separated by solid walls, the conditions of dynamic equilibrium must be satisfied in directions normal to the flow as well as along the channel. If the flow at infinite distance from the body is parallel and uniform and the effect of viscosity can be neglected, these equilibrium conditions are identically satisfied by the existence of a velocity potential, *i.e.*, by the fact that the flow is irrotational in the whole field.

Using Cartesian coordinates x_1, x_2, x_3, and denoting the velocity components by v_1, v_2, v_3, the continuity equation which corresponds to Eq. (17) can be written in the form

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho v_i) = 0 \tag{21}$$

For stationary flow $\partial\rho/\partial t = 0$ and therefore

$$\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} + \frac{1}{\rho} \sum_{i=1}^{3} v_i \frac{\partial \rho}{\partial x_i} = 0 \tag{22}$$

For irrotational flow it follows from Bernoulli's equation that for $i = 1, 2$ and 3

$$\frac{\partial p}{\partial x_i} = \frac{dp}{d\rho}\frac{\partial \rho}{\partial x_i} = a^2 \frac{\partial \rho}{\partial x_i} = -\frac{\rho}{2}\frac{\partial v^2}{\partial x_i} \tag{23}$$

where $v^2 = v_1^2 + v_2^2 + v_3^2$. Substituting the value of $\partial\rho/\partial x_i$ from Eq. (23) in Eq. (22) the following relation is obtained:

$$\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = \frac{1}{a^2} \sum_{i=1}^{3}\sum_{k=1}^{3} v_i v_k \frac{\partial v_i}{\partial x_k} \tag{24}$$

Introducing the velocity potential φ, defined by the relations $v_i = \partial\varphi/\partial x_i$, one obtains

$$\Delta\varphi = \sum_{i=1}^{3} \frac{\partial^2 \varphi}{\partial x_i^2} = \frac{1}{a^2} \sum_{i=1}^{3}\sum_{k=1}^{3} \frac{\partial \varphi}{\partial x_i}\frac{\partial \varphi}{\partial x_k}\frac{\partial^2 \varphi}{\partial x_i \partial x_k} \tag{25}$$

It is seen that in the case of an incompressible fluid ($a = \infty$) the right side of the equation vanishes and φ is determined by the equation of Laplace. If a is finite, the equation is linear in the second derivatives of the potential, but the coefficients are functions of the velocity components and the velocity of sound, which it-

self is a function of the magnitude of the resulting velocity according to the relation (6):

$$a^2 = a_0^2 - [(\gamma - 1)/2]v^2 \tag{26}$$

Eq. (25) applied to the two-dimensional problem, with the notations $x_1 = x$, $x_2 = y$, $v_1 = v_x$, $v_2 = v_y$, can be written in the form

$$\left(1 - \frac{v_x^2}{a^2}\right)\frac{\partial^2\varphi}{\partial x^2} - 2\frac{v_x v_y}{a^2}\frac{\partial^2\varphi}{\partial x \partial y} + \left(1 - \frac{v_y^2}{a^2}\right)\frac{\partial^2\varphi}{\partial y^2} = 0 \tag{27}$$

It appears that—with the exception of some cases which have very simple symmetry conditions—it is extremely difficult to obtain exact solutions of the complicated non-linear differential Eqs. (25) or (27). In the following sections methods for approximate solution of these differential equations are discussed.

Expansion in Powers of the Mach Number

Assume that the velocity potential is expanded in a power series of powers of the square of the Mach number M in the form:

$$\varphi = U[\Phi_0 + \Phi_1 M^2 + \Phi_2 M^4 + \ldots.] \tag{28}$$

where U is the velocity of the undisturbed flow, $M = U/a$, and $\Phi_0, \Phi_1, \ldots.$ are functions of x_1, x_2, x_3.

Substituting (28) in Eq. (25) one obtains for the coefficient of the zero approximation the equation

$$\Delta\Phi_0 = 0 \tag{29}$$

i.e., the equation for incompressible fluids. To obtain the coefficient of the next approximation Φ_1 the right side of Eq. (25) has to be expanded in powers of M^2, taking into account the expression (26) for a^2. Then one obtains

$$\Delta\Phi_1 = \sum_{i=1}^{3}\sum_{k=1}^{3}\frac{\partial\Phi_0}{\partial x_i}\frac{\partial\Phi_0}{\partial x_k}\frac{\partial^2\Phi_0}{\partial x_i \partial x_k} \tag{30}$$

In a more physical language, after solving the flow problem for incompressible fluids, the terms on the right side of Eq. (30) are being computed using this solution and are being substituted in the equation. Therefore Eq. (30) has the form

$$\Delta\Phi_1 = F(x_1, x_2, x_3) \tag{31}$$

of a non-homogeneous linear differential equation. If Φ_1 is known, an equation of similar form is obtained for Φ_2, and so on for the higher coefficients.

The problem of solving Eq. (31) for the given boundary conditions at the surface of the body is identical with the problem of finding the flow of an imcompressible fluid around the body produced by a field of continuously distributed sources of the intensity $F(x_1, x_2, x_3)$ outside of the body. The flow produced by a source around bodies of certain simple shapes like, for example, circular or elliptical cylinders, sphere or ellipsoids is known and therefore in such cases the solution of Eq. (30) can be obtained by superposition. However, the procedure is very laborious, especially for the higher coefficients Φ_2, Φ_3, etc., and if $M > 0.4$ the series (28) converges very slowly.

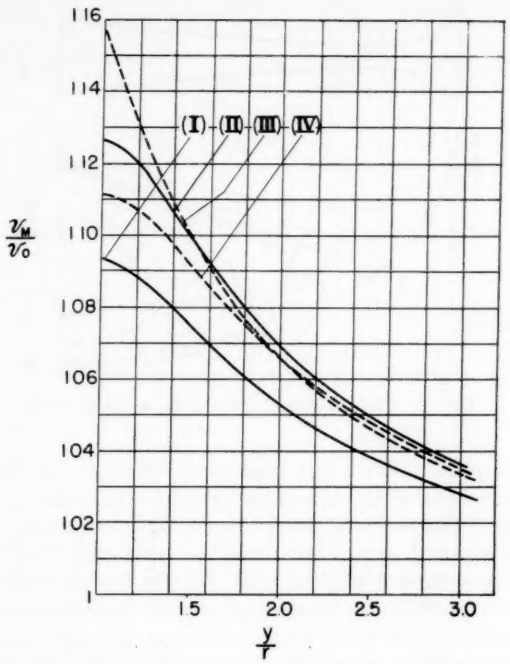

Fig. 5. The velocity correction factors v_M/v_0 for flow around a circular cylinder.
(I) First approximation.
(II) Second approximation, $\gamma = 1.405$.
(III) Hodograph method, $\gamma = -1$, distortion uncorrected.
(IV) Hodograph method, $\gamma = -1$, distortion corrected.

The method of expansion—which can be also considered as an iteration method—was used by Janzen and Lord Rayleigh for computation of the flow of a compressible fluid around a circular cylinder; then by I. Imai, and by K. Tamada and Y. Saito.

For the two-dimensional case Imai and Aihara gave an elegant method for finding a particular integral of the non-homogeneous equation for the first approximation:

$$\Delta\Phi_1 = \left(\frac{\partial\Phi_0}{\partial x}\right)^2\frac{\partial^2\Phi_0}{\partial x^2} + 2\frac{\partial\Phi_0}{\partial x}\frac{\partial\Phi_0}{\partial y}\frac{\partial^2\Phi_0}{\partial x \partial y} + \left(\frac{\partial\Phi_0}{\partial y}\right)^2\frac{\partial^2\Phi_0}{\partial y^2} \tag{32}$$

It is known that if $\Phi_0(x, y)$ is a solution of Laplace's equation, Φ_0 is the real part of the complex function $F = \Phi_0 + i\Psi_0$ of the complex variable $z = x + iy$. Introduce the complex conjugate of F; namely, $\bar{F} = \Phi_0 - i\Psi_0$ and the conjugate of z; namely, $\bar{z} = x - iy$. Then the right side of Eq. (32) can be written in the simple form $R(d^2\bar{F}/d\bar{z}^2)(dF/dz)^2$ and it can be shown that the real part of the complex quantity $(d\bar{F}/d\bar{z})\int(dF/dz)^2 dz$ is a particular integral of Eq. (32). To find

the complete solution one has to add an integral of the homogeneous equation $\Delta\Phi_1 = 0$ which together with the particular integral obtained satisfies the boundary conditions.

It is seen that provided a closed-form solution of the flow problem for incompressible fluids is known, as, for example, in the case of a circular or elliptical cylinder or a Joukowsky airfoil, the first approximation for a compressible fluid can be obtained by direct integration and solution of Laplace's equation for given boundary values.

Fig. 5 refers to the flow around a circular cylinder. It gives the correction factor to be applied to the velocity distribution over a section passing through the center of the circle normal to the main flow direction; y is the distance from the center, r the radius of the cylinder. The correction factor plotted in the diagram is v_M/v_0, where v_M refers to $M = 0.4$ and v_0 to the solution for incompressible fluids. The two curves (I) and (II) show the first and second approximations obtained by Imai and Tamada by using the method of sources (developed by L. Poggi). The value $\gamma = 1.405$ is used for the second approximation; the first approximation is independent of the value of γ. The rest of the curves plotted in the figure will be discussed in the section on Applications of the Hodograph Method.

The following table shows the particular cases treated by means of the method of expansion up to the present.

	First Approximation	Second Approximation
Circular cylinder	Lord Rayleigh, O. Janzen, L. Poggi	I. Imai, K. Tamada and Y. Saito
Elliptic cylinder	S. G. Hooker, C. Kaplan, I. Imai and T. Aihara	
Joukowsky airfoil	L. Poggi, C. Kaplan	
Sphere	K. Tamada	K. Tamada

Method of Small Perturbations

It is known that the so-called thin airfoil theory has been applied with success to airfoils of small thickness. This theory is based on the assumption that the velocity changes caused in a uniform parallel airstream by the presence of the airfoil are small in comparison with the velocity of the undisturbed flow.

The same assumption was applied by Glauert, Ackeret and Prandtl to the flow of a compressible fluid around airfoils or slender bodies. Denote the velocity of the undisturbed flow by V and assume that the squares and products of the quantities $(v_1 - V)$, v_2 and v_3 can be neglected. Then Eq. (24) obtains the form

$$\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = \frac{V^2}{a^2} \frac{\partial v_1}{\partial x_1} \tag{33}$$

The local velocity of sound is determined by the relation of Eq. (6).

$$a^2 + \frac{\gamma - 1}{2} v^2 = a_1^2 + \frac{\gamma - 1}{2} V^2 \tag{34}$$

It is seen that the difference between a^2 and a_1^2, *i.e.*, between the squares of the local velocity of sound and the velocity of sound in the undisturbed fluid can be neglected in this approximation. With the introduction of the velocity potential in the form $\varphi = Vx + \varphi'$ Eq. (33) furnishes the following linear differential equation for the "perturbation potential" φ':

$$\left(1 - \frac{V^2}{a_1^2}\right) \frac{\partial^2 \varphi'}{\partial x_1^2} + \frac{\partial^2 \varphi'}{\partial x_2^2} + \frac{\partial^2 \varphi'}{\partial x_3^2} = 0 \tag{35}$$

This equation can be reduced to Laplace's equation provided $V < a_1$, by introducing as new independent variables $\xi_1 = x_1$, $\xi_2 = x_2\sqrt{1 - M_1^2}$ and $\xi_3 = x_3\sqrt{1 - M_1^2}$, where $M_1 = V/a_1$ is the Mach number for the undisturbed flow. If $(V/a_1) > 1$, *i.e.*, the undisturbed flow is supersonic, Eq. (35) has the character of the wave equation.

From Eq. (35) it is seen that the potential function for a flow in an incompressible fluid around a surface whose equation is given by $x_2 = f(x_1)$, $x_3 = g(x_1)$ can be expressed approximately in the form

$$\varphi(x_1, x_2, x_3) = Vx_1 + \varphi'(x_1, x_2\sqrt{1 - M_1^2}, x_3\sqrt{1 - M_1^2}) \tag{36}$$

where φ' is a solution of the Laplace equation for the variables $\xi_1 = x_1$, $\xi_2 = x_2\sqrt{1 - M_1^2}$, $\xi_3 = x_3\sqrt{1 - M_1^2}$. This result can be interpreted in the following way.

Evidently

$$\varphi(\xi_1, \xi_2, \xi_3) = V\xi_1 + \varphi'(\xi_1, \xi_2, \xi_3)$$

represents a flow of an incompressible fluid in the ξ_1, ξ_2, ξ_3 coordinate system. One has then to investigate the shape of the body in this system by comparison of the streamlines in the x_1, x_2, x_3 and the ξ_1, ξ_2, ξ_3 coordinate system. For sake of simplicity take a body with axial symmetry and compute the inclination β of an arbitrary streamline in the $x_1\, x_2$ plane. With the approximation used in this section $\tan \beta = (1/V)(\partial\varphi'/\partial x_2)$ in the x_1, x_2, x_3 system and $\tan \beta' = (1/V)(\partial\varphi'/\partial\xi_2)$ in the ξ_1, ξ_2, ξ_3 system. It is seen that

$$\frac{\tan \beta'}{\tan \beta} = \frac{\partial\varphi'}{\partial\xi_2} \Big/ \frac{\partial\varphi'}{\partial x_2} = \frac{1}{\sqrt{1 - M_1^2}} \tag{37}$$

From Eq. (37) it follows that the flow in the ξ_1, ξ_2, ξ_3 system represents the flow around a body obtained from the original body by expanding its lateral dimension in the ratio $1:\sqrt{1 - M_1^2}$.

Therefore, if one has to determine the flow of a compressible fluid around a slender body, the Mach number of the undisturbed flow being equal to M_1, this can be accomplished approximately by expanding the dimensions of the body normal to the main flow in the ratio $1:\sqrt{1 - M_1^2}$ and solving the problem of the flow of an incompressible fluid around the body of the expanded shape. Especially it can be shown that with the ap-

proximation used in this section the pressures acting on corresponding points of the two bodies are equal.

Namely neglecting the squares and products of perturbance velocity components considered small in comparison with V the pressure at an arbitrary point of a compressible or incompressible fluid is equal to

$$p - p_1 = -\rho_1 V(v_1 - V) = -\rho_1 V v_1' \qquad (38)$$

where v_1' denotes the perturbation velocity in the direction of the main flow and p_1 refers to the undisturbed flow. Since $(\partial\varphi'/\partial x_2) = (\partial\varphi'/\partial\xi_2)$, *i.e.*, the v_1' components are equal in the two cases, the pressures are also approximately equal.

Compare now the pressure distribution acting in an incompressible fluid on two slender bodies, assuming that the shape of the second is obtained by lateral expansion in the ratio $1:n$. If the potential function for the flow around the first body is

$$\varphi = V\xi_1 + \varphi'(\xi_1, \xi_2, \xi_3)$$

the potential function for the flow around the second body is approximately given by

$$\varphi = V\xi_1 + n\varphi'(\xi_1, \xi_2, \xi_3) \qquad (39)$$

This can be shown by consideration of the inclination of the streamlines in the manner indicated above. Then it is seen from Eq. (38) that the perturbation velocities v_1' and the pressure differences $p - p_1$ are also in the ratio $1:n$.

This result can be applied to an approximate calculation of the pressure coefficient in a compressible fluid if the value of the pressure coefficient for the flow of an incompressible fluid around the same body is known. It was shown that the pressure acting in a compressible fluid is equal to the pressure acting in an incompressible fluid on a body obtained by expansion in the ratio $1:\sqrt{1 - M_1^2}$. The incompressible flow around this body can now be compared with the incompressible flow around the original body. As it was explained above, the pressure coefficients vary proportionally with the lateral dimensions; hence the pressure coefficients at corresponding points are in the ratio $1:\sqrt{1 - M_1^2}$. Therefore, if the pressure coefficient at an arbitrary point of a body in an incompressible fluid is equal to $C_{p_0} = \Delta p/(\rho/2)V^2$, the pressure coefficient in a compressible fluid is approximately equal to $C_{pM} = (1/\sqrt{1 - M_1^2})C_{p_0}$.

This relation of course cannot be correct for the neighborhood of the stagnation point, where the perturbance velocity is of the same order as the undisturbed velocity V. Nevertheless, if the influence of such regions is small, the same rule applies to the total forces acting on the body. For example, as was shown by Glauert, Ackeret and Prandtl, the lift coefficient of an airfoil moving with high velocity can be obtained from the lift coefficient obtained for small values of M_1 by the formula

$$C_{L_M} = (1/\sqrt{1 - M_1^2})C_{L_0} \qquad (40)$$

In the case of an airfoil the "expansion" normal to the main flow is essentially equivalent to a change of angle of attack, camber and thickness.

According to Eq. (40) the slope of the lift curve $dC_L/d\alpha$ should also vary in proportion to $1/\sqrt{1 - M_1^2}$. However, the comparison with experimental results does not permit one to formulate convincing final conclusions. In some cases good agreement is found; in other cases the rule is at variance with the experimental evidence. It is the author's impression that in most cases the Glauert-Prandtl rule underestimates the compressibility effect on the slope of the lift curve. For example, in Fig. 6 the ratio $(dC_L/d\alpha)_M/(dC_L/d\alpha)_0$ is plotted as a function of M_1. The experimental points

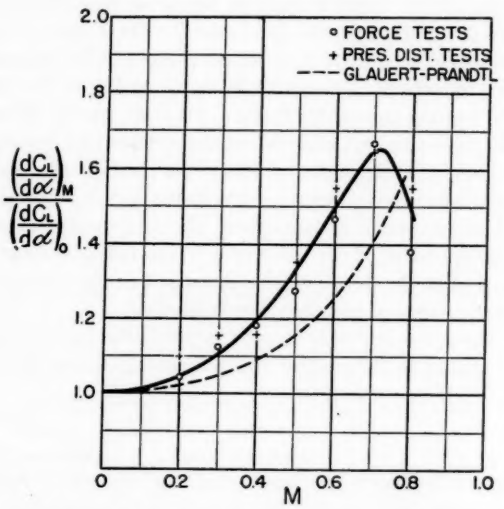

FIG. 6. Relation between lift-curve slope $dC_L/d\alpha$ and Mach number M, for N.A.C.A. 4412 (J. Stack, W. F. Lindsey, R. E. Littell).

are computed from Fig. 9 of N.A.C.A. Technical Report No. 646 and refer to the N.A.C.A. 4412 section. The value for $(dC_L/d\alpha)_0$ is taken from the pressure distribution tests at a low Mach number. Fig. 7 shows the values of the same ratio for a propeller section of 6.32 per cent thickness investigated in the high-speed tunnel at Guidonia.

The slope $dC_L/d\alpha$ reaches a peak value at a Mach number between about 0.6 and 0.8; beyond this peak a more or less rapid drop takes place. This drop apparently coincides with the first occurrence of a shock wave. This effect will be discussed in a later section.

THE HODOGRAPH METHOD

The analytical method presented in this section is restricted to two-dimensional problems. In the section on One-Dimensional Gasdynamics the fundamental differential equation for the two-dimensional irrotational motion of a compressible fluid in Cartesian coordinates is given. In many problems it is convenient to use the arc length along streamlines, and equipotential lines as

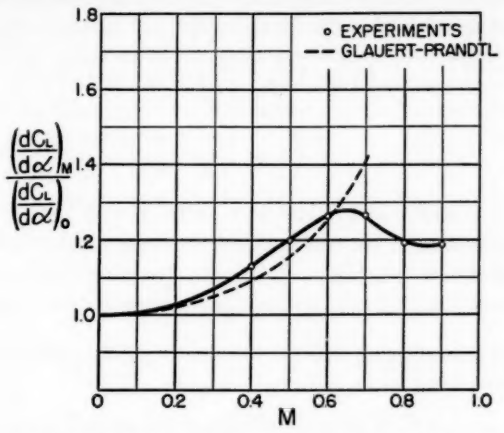

FIG. 7. Relation between lift-curve slope $dC_L/d\alpha$ and Mach number M, for a 6 per cent thick Italian propeller section (A. Ferri).

independent variables; the magnitude of the velocity v and the angle of inclination between the velocity vector and a fixed direction (*e.g.*, the direction of the undisturbed flow) as unknown functions. Then the velocity

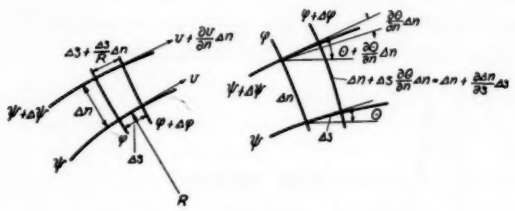

FIG. 8.

distribution is determined by the conditions of irrotationality and continuity. The rotation of the fluid in the case of curved flow is given by (Fig. 8)

$$\omega = (\partial v/\partial n) + (v/R) \tag{41}$$

where v is the magnitude of the velocity, n is the arc length taken along an equipotential line, and R is the radius of curvature of the streamline. Evidently $1/R = \partial\theta/\partial s$, where s is the arc length taken along a streamline. Hence, one obtains as the condition of irrotationality:

$$\frac{1}{v}\frac{\partial v}{\partial n} - \frac{\partial \theta}{\partial s} = 0 \tag{42}$$

The continuity condition requires that for an elementary channel between two neighboring streamlines $\rho v \Delta n = \text{const.}$, *i.e.*,

$$\frac{1}{\rho}\frac{\partial \rho}{\partial s} + \frac{1}{v}\frac{\partial v}{\partial s} + \frac{1}{\Delta n}\frac{\partial \Delta n}{\partial s} = 0 \tag{43}$$

According to Eq. (19)

$$\frac{1}{\rho}\frac{\partial \rho}{\partial s} + \frac{1}{v}\frac{\partial v}{\partial s} = (1 - M^2)\frac{1}{v}\frac{\partial v}{\partial s} \tag{44}$$

On the other hand, it is seen from Fig. 8 that

$$\frac{1}{\Delta n}\frac{\partial \Delta n}{\partial s} = \frac{\partial \theta}{\partial n} \tag{45}$$

Hence one obtains the equation

$$(1 - M^2)\frac{1}{v}\frac{\partial v}{\partial s} + \frac{\partial \theta}{\partial n} = 0 \tag{46}$$

The two Eqs. (42) and (46) determine the velocity distribution in the following sense. Assume one draws arbitrarily an orthogonal system of curves $\varphi = \text{const.}$, $\psi = \text{const.}$ Then by integrating Eqs. (42) and (46), for example, by a step by step method, one obtains certain values for v and θ. Obviously the values of the angle θ obtained in this way should be consistent with the arbitrarily assumed streamlines, θ indicating the direction of the velocity vector and therefore the streamline at an arbitrary point. If this is not the case, one has to correct the network and continue the procedure which is essentially a trial and error method. The Russian mathematician, Chaplygin, noticed that the problem can be greatly simplified if a change of variables is performed by using v and θ as independent variables and φ and ψ as unknown functions. His method amounts essentially to plotting a network $\varphi = \text{const.}$, $\psi = \text{const.}$ in a plane in which the velocity components v_x and v_y are the Cartesian coordinates or v and θ the polar coordinates. In other words, as one proceeds along a streamline or an equipotential line in the "physical plane" (the plane in which the flow actually takes place), one plots (Fig. 9) the velocity vector as to magnitude and direction starting from the origin in a second

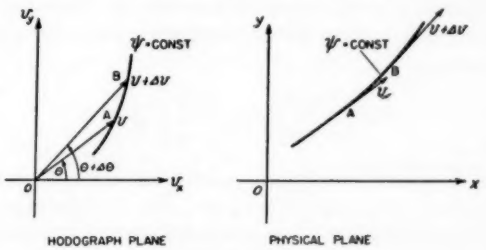

FIG. 9.

plane, the "hodograph plane." The curves described by the end points of the velocity vector are considered as the curves $\psi = \text{const.}$ and $\varphi = \text{const.}$, respectively.

It is important to obtain the equations for φ and ψ as functions of the new independent variables v and θ. The first step consists of introducing in Eqs. (42) and (46) φ and ψ as independent avariables instead of s and n by means of the relations $d\varphi = vds$ and $d\psi = \rho v dn$. Then Eqs. (42) and (46) obtain the form

$$\frac{\rho}{v}\frac{\partial v}{\partial \psi} - \frac{\partial \theta}{\partial \varphi} = 0$$

$$(1 - M^2)\frac{1}{v}\frac{\partial v}{\partial \varphi} + \rho\frac{\partial \theta}{\partial \psi} = 0 \tag{47}$$

Now the independent and dependent variables have to be interchanged. This is a well-known operation in mathematical analysis. If y is a function of the independent variable x and one chooses to consider x as a function of y, obviously from $dx/dx = (dx/dy)(dy/dx) = 1$ follows $dx/dy = 1/(dy/dx)$. Using the obvious identities: $\partial\varphi/\partial\varphi = \partial\psi/\partial\psi = 1, \partial\varphi/\partial\psi = \partial\psi/\partial\varphi = 0$, the following analogous relations between the partial differential coefficients are obtained:

$$\frac{\partial v}{\partial \psi} = \frac{1}{\Delta}\frac{\partial \varphi}{\partial \theta}, \qquad \frac{\partial \theta}{\partial \psi} = -\frac{1}{\Delta}\frac{\partial \varphi}{\partial v}$$

$$\frac{\partial v}{\partial \varphi} = -\frac{1}{\Delta}\frac{\partial \psi}{\partial \theta}, \qquad \frac{\partial \theta}{\partial \varphi} = \frac{1}{\Delta}\frac{\partial \psi}{\partial v} \tag{48}$$

where Δ is the so-called "Jacobian"; defined by

$$\Delta = \frac{\partial \varphi}{\partial v}\frac{\partial \psi}{\partial \theta} - \frac{\partial \varphi}{\partial \theta}\frac{\partial \psi}{\partial v} \tag{49}$$

The expression Δ is denoted in general by $\partial(\varphi, \psi)/\partial(v, \theta)$ and has to some extent properties analogous to a differential quotient. For example, the product of differentials $d\varphi d\psi$ is equal to $(\partial(\varphi, \psi)/\partial(v, \theta))dvd\theta$, just as the differential dy is equal to $(dy/dx)dx$.

Introducing the expression (48) in Eq. (47) one obtains:

$$\frac{\rho}{v}\frac{\partial \varphi}{\partial \theta} - \frac{\partial \psi}{\partial v} = 0$$

$$(1 - M^2)\frac{1}{\rho v}\frac{\partial \psi}{\partial \theta} + \frac{\partial \varphi}{\partial v} = 0 \tag{50}$$

Eliminating φ the calculation results in obtaining the following linear partial differential equation for ψ as a function of v and θ:

$$\frac{1 - M^2}{\rho^2}\frac{\partial^2 \psi}{\partial \theta^2} + \left(\frac{v}{\rho}\frac{\partial}{\partial v}\right)^2 \psi = 0 \tag{51}$$

It is evident that since v and θ are the independent variables, and ρ and M are functions of v, this equation is a linear differential equation with variable coefficients, but the variable coefficients are functions of the independent variables only whereas in the case of Eq. (27) the coefficients are functions of the derivative of the unknown variable. Therefore, the solution of the Eq. (27) can be obtained only by a laborious iteration process, whereas the methods of analysis can be applied to Eq. (51) for obtaining families of exact solutions. This does not mean that one is able to solve exactly the flow problem for given boundary conditions in the physical plane, but one obtains exact solutions in the hodograph plane, which can be transferred to the physical plane and furnish certain exact flow patterns of a compressible fluid.

Chaplygin and his followers noticed that Eq. (51) is reduced to Laplace's equation if it is assumed that $(1 - M^2)/\rho^2$ is a constant quantity. Using the general thermodynamic relations, Eqs. (6) and (7), this expression is obtained in the form

$$\frac{1 - M^2}{\rho^2} = \frac{1}{\rho_0^2}\left(1 - \frac{\gamma + 1}{2}\frac{v^2}{a_0^2}\right) \times \left(1 - \frac{\gamma - 1}{2}\frac{v^2}{a_0^2}\right)^{(1+\gamma)/(1-\gamma)} \tag{52}$$

It is seen that $1 - M^2/\rho^2 = \text{const.}$ if $\gamma = -1$. For this value of γ the relation between density and velocity is given by Eq. (7)

$$\rho = \rho_0[1 + (v^2/a_0^2)]^{-1/2} \tag{53}$$

and according to Eq. (6) the square of the velocity of sound is equal to

$$a^2 = a_0^2[1 + (v^2/a^2)] = a_0^2\,(\rho_0^2/\rho^2) \tag{54}$$

Now using the definition of a^2

$$a^2 = dp/d\rho = a_0^2(\rho_0^2/\rho^2) \tag{55}$$

one obtains by integration the pressure-density relation

$$p = \text{const.} - (a_0^2\rho_0^2/\rho) \tag{56}$$

This relation, of course, is not satisfied by any real gas, since for all real gases $1 < \gamma < 1.66$, however, the constant of integration in Eq. (53) can be determined in such a way that in the p, $1/\rho$ diagram (Fig. 10) the

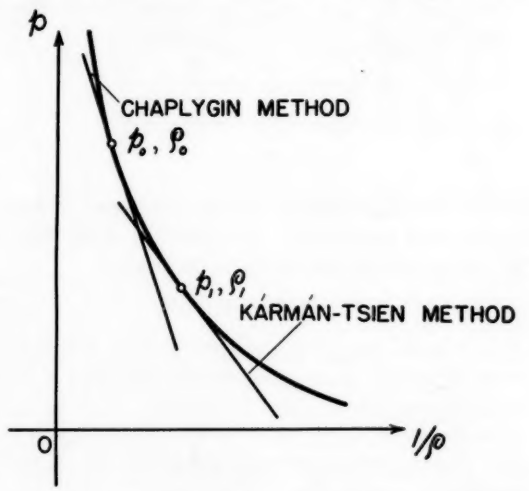

FIG. 10. Approximations to true isentropic pressure-volume curve.

straight line defined by Eq. (53) is tangent to the actual isentropic curve of the real gas. Chaplygin determined the constant so that the straight line approximates the isentropic curve near the point corresponding to the state of the gas in rest. In this way he and his followers solved certain problems by integration of Eq. (51) and obtained good approximations if the maximum velocity did not exceed about 0.4 times the velocity of sound.

In order to apply the hodograph method to larger velocities one can determine the constant of integration

in Eq. (55) so that the isentropic curve of the gas is approximated near the point p_1, $1/\rho_1$ which corresponds to the undisturbed flow with velocity V. This idea was carried out at the suggestion of the present author by H. S. Tsien, and will be discussed in the following section.

Applications of the Hodograph Method

If the quantity $(1 - M^2)/\rho^2$ is assumed constant, after division of Eq. (51) by $(1 - M^2)/\rho^2$, this constant factor can be included in the bracket and one obtains

$$\frac{\partial^2\psi}{\partial\Theta^2} + \left(\frac{v}{\sqrt{1 - M^2}}\frac{\partial}{\partial v}\right)^2\psi = 0 \tag{57}$$

Assume that w, Θ denote the magnitude and the angle of inclination of the velocity vector of an incompressible fluid. Then for an arbitrary flow pattern of this incompressible fluid the stream function satisfies the equation

$$\partial^2\psi/\partial\Theta^2 + [w(\partial/\partial w)]^2\psi = 0 \tag{58}$$

Consequently, if $\psi(w, \Theta)$ is a known flow pattern of an incompressible fluid and w and v are connected by the relation

$$\sqrt{1 - M^2}(dv/v) = (dw/w) \tag{59}$$

the function $\psi(w(v), \Theta)$ determines the flow of a compressible fluid characterized by the pressure-density ratio (56). The factor $\sqrt{1 - M^2}$ can be expressed by the local velocity v, the velocity V and the Mach number M_1 of the undisturbed flow. Taking into account the relation

$$a^2 - v^2 = a_1^2 - V^2$$

where a_1 is the velocity of sound in the undisturbed flow, one obtains

$$M^2 = \frac{v^2}{a^2} = \frac{v^2}{a_1^2 - V^2 + v^2} = \frac{M_1^2(v/V)^2}{1 - M_1^2 + M_1^2(v/V)^2}$$

and

$$\sqrt{1 - M^2} = \sqrt{1 - M_1^2}/\sqrt{1 - M_1^2 + M_1^2(v/V)^2}$$

Hence, the relation between the velocities of the incompressible and compressible fluids is given by

$$dw/w = \sqrt{1 - M_1^2}/\sqrt{1 - M_1^2 + M_1^2(v/V)^2}(dv/v) \tag{60}$$

By integrating and putting $w\ (V) = W$ one obtains

$$\frac{w}{W} = \frac{v}{V}\frac{1 + \sqrt{1 - M_1^2}}{\sqrt{1 - M_1^2 + M_1^2(v/V)^2} + \sqrt{1 - M_1^2}}$$

or

$$v/V = (w/W)(1 - \lambda)/1 - \lambda(w/W)^2 \tag{61}$$

where

$$\lambda = M_1^2/(1 + \sqrt{1 - M_1^2})^2$$

The pressure coefficient corresponding to the Mach number $M_1 = V/a_1$ is given by

$$C_{p_M} = \frac{2(p - p_1)}{\rho_1 V_1^2} = C_{p_0}\frac{1}{\sqrt{1 - M_1^2} + \dfrac{M_1^2}{1 + \sqrt{1 - M_1^2}}\dfrac{C_{p_0}}{2}} \tag{62}$$

provided C_{p_0} is the value of the pressure coefficient for $M_1 = 0$. It is seen that Eq. (62) differs from the Glauert-Prandtl formula by the second term in the denominator which itself contains C_{p_0} again. Hence, if C_{p_0} is negative, the magnitude of the compressibility effect predicted by Eq. (62) is greater than that predicted by the Glauert-Prandtl formula. The formula (62) can be applied to calculation of the influence of compressibility on the pressure distribution of airfoils, if the pressure distribution at low Mach numbers is known. In this application a slight distortion of the shape of the airfoil is disregarded; namely, if $\psi(w, \Theta)$ is the stream function for the flow of an incompressible fluid around a given section, so that, for example, the streamline $\psi = 0$ corresponds to the boundary of the section, then the streamline $\psi = 0$ of the corresponding compressible flow in the physical plane will not coincide exactly with the boundary of the original section. It appears—at least in an example investigated by Tsien and discussed later in this section—that the distortion of the section is relatively small (with the exception of such thick sections as a circle) and the error made in the pressure coefficient tends to correct the error made by use of the approximate pressure-density relation.

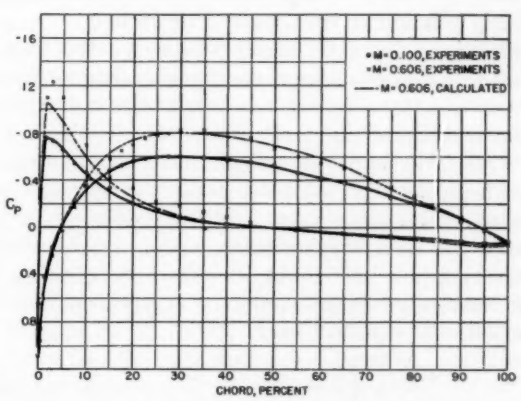

FIG. 11. Pressure distributions for N.A.C.A. 4412 at $\alpha = -2°$.

In Fig. 11 the measured pressure distributions of the N.A.C.A. 4412 airfoil are plotted for $M_1 = 0.100$ and $M_1 = 0.606$ according to N.A.C.A. Technical Report No. 646. The dotted line is calculated for $M_1 = 0.606$

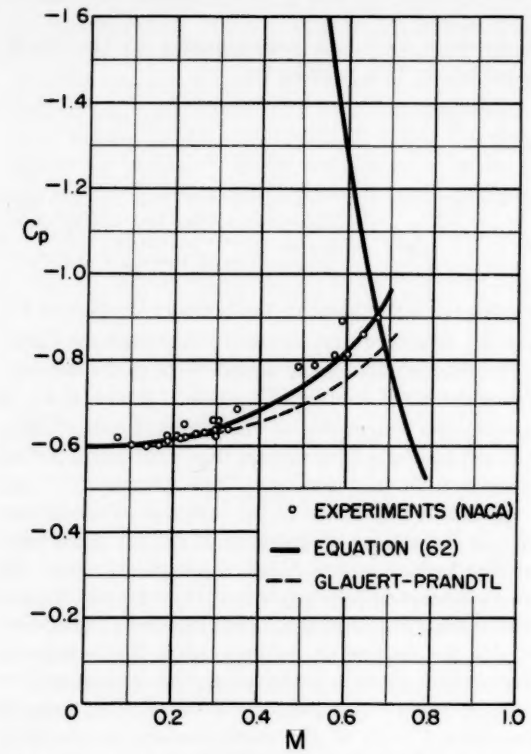

FIG. 12 (a). Increase of the pressure coefficient C_p with Mach number M, for N.A.C.A. 4412.
At 30 per cent chord, upper surface, $\alpha = -2°$.

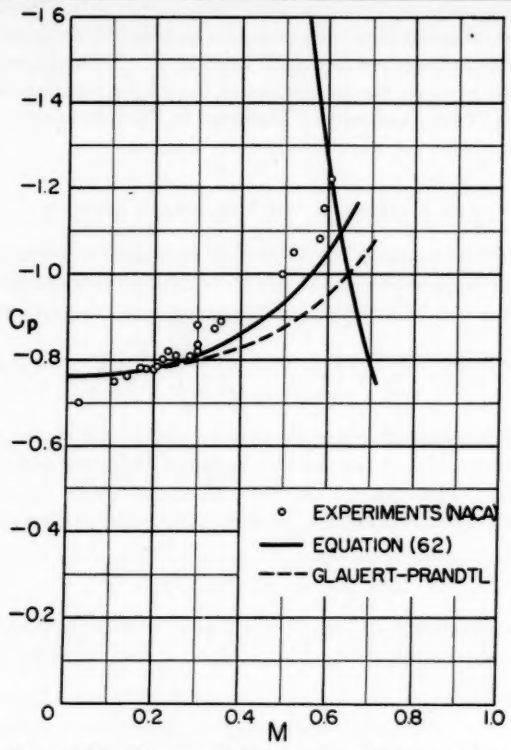

FIG. 12 (b). Increase of the pressure coefficient C_p with Mach number M, for N.A.C.A. 4412.
At 27.5 per cent chord, upper surface, $\alpha = -0°15'$.

by Eq. (62) using for C_{p_0} the values measured at $M_1 = 0.100$ and $\alpha = -2$ degrees.

Figs. 12 (a, b, c) show the variation of pressure coefficients at three points of the same airfoil section; the measured values are compared with the values computed by Eq. (62) and by the Glauert-Prandtl rule.

Strictly speaking, the theory developed in this section deals with a specific kind of compressible fluid for which $\gamma = -1$. However, one can look at the method also from a different point of view. For air, the factor $(1-M^2)/\rho^2$ in Eq. (51) is variable, but a fair approximation might be expected if this variable coefficient is replaced by its mean value. This is probably correct unless the local Mach number comes close to unity. Replacing $(1 - M^2)/\rho^2$ by its value corresponding to the undisturbed fluid, *i.e.*, by $(1 - M_1^2)/\rho_1^2$, one obtains the following equation for ψ:

$$\frac{\partial^2\psi}{\partial\Theta^2} + \left(\frac{v}{\rho}\frac{\rho_1}{\sqrt{1 - M_1^2}}\frac{\partial}{\partial v}\right)^2\psi = 0 \qquad (63)$$

Then Eq. (59) is replaced by

$$\sqrt{1 - M_1^2}\,\frac{\rho}{\rho_1}\frac{dv}{v} = \frac{dw}{w} \qquad (64)$$

Eq. (64) leads to the following approximate relation between v and w:

$$(w/W)^k = (v/V)e^{-0.230(V/a_0)^2[(v^2/V^2)-1]} \qquad (65)$$

where

$$k = \left(1 + \frac{\gamma - 1}{2}M_1\right)^{-1/(\gamma-1)}/\sqrt{1 - M_1^2}$$

The pressure coefficients calculated from (65) are shown in Fig. 13; they are only slightly different from those obtained from Eq. (62) plotted in the same figure by dotted lines. For larger values of M_1 Eq. (65) apparently gives better agreement with the experiments.

The physical meaning of the Eqs. (63) and (57) can be described in the following way. The stream function $\psi[w(v), \Theta]$ which is a solution of Eq. (63) furnishes a family of approximate streamlines for a compressible fluid satisfying the exact relations between pressure, density and velocity; whereas the stream function satisfying Eq. (57) furnishes exact streamlines of a compressible fluid whose physical behavior is only approximately correct. The second method has the advantage of simplicity in the mathematical expressions.

In order to have an estimate of the error made by use of the approximate pressure-density relation, a calculation was made in an extreme case; namely, in the case of a circular cylinder. Fig. 5, which was partly discussed before in the section on Expansion in Powers on the Mach Number, shows the velocity correction factors

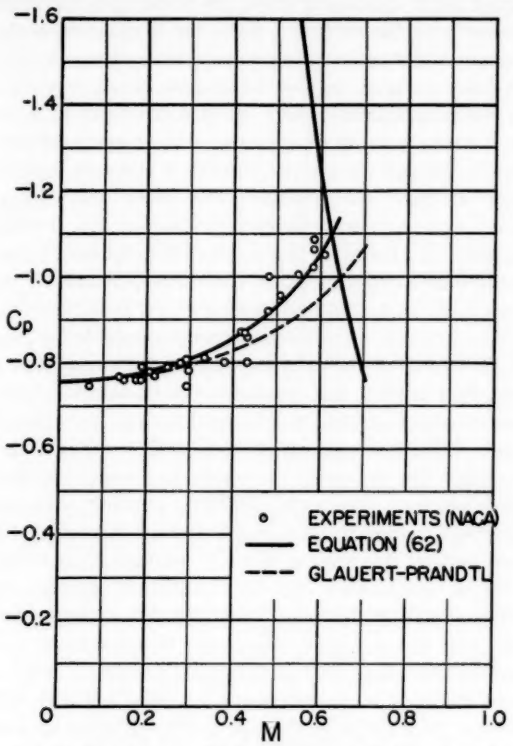

FIG. 12 (c). Increase of the pressure coefficient C_p with Mach number M, for N.A.C.A. 4412.
At 2.5 per cent chord, lower surface, $\alpha = -2°$.

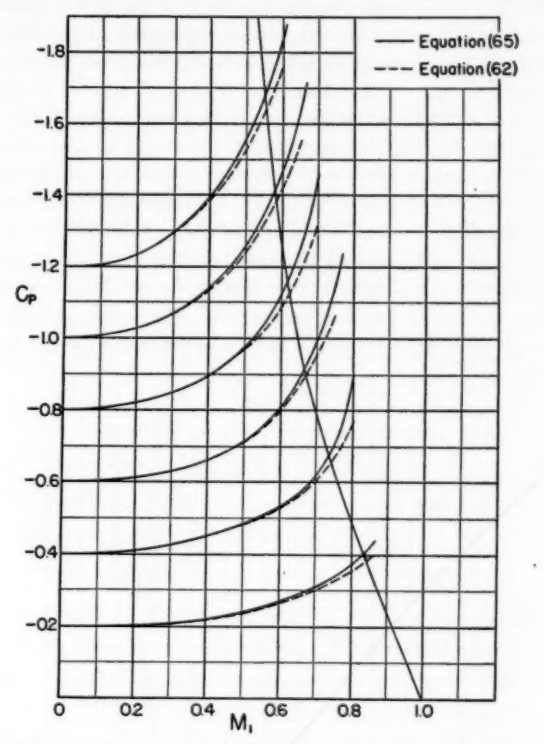

FIG. 13. Pressure coefficients C_p *vs.* Mach number calculated from Eqs. (62) and (65).

v_M/v_0 for $M_1 = 0.4$ calculated under two different assumptions. Curve III is obtained without correction of the distortion of the original section, Curve IV by solving the flow pattern for an originally elliptical cylinder which by the distortion becomes almost exactly circular (within $^1/_{50}$ of a per cent of the radius). In both cases the approximate pressure-density relation (56) is employed, which makes an exact solution of the flow equations possible. These curves can be compared with the second approximation obtained for $\gamma = 1.405$ (Curve II). This curve probably lies somewhat below the curve which would represent the exact solution for $\gamma = 1.405$. It is seen that the error made by the distortion lies in the right direction to balance the error made by using $\gamma = -1$ instead of the correct value of γ.

DRAG AND TRANSITION THROUGH THE VELOCITY OF SOUND

The experimental evidence shows that the compressibility effect on the drag of airfoil sections can be attributed to two different physical processes. At low values of the Mach number the drag of most airfoil sections consists essentially of skin friction of partly laminar and to a greater part turbulent character. The compressibility effect on the skin friction coefficient of a boundary layer with a fixed pressure gradient is very small in the range of the Mach numbers considered. However, with increasing Mach number the local velocity at the surface increases faster than in the undisturbed stream and this causes an increase of the skin friction. Another effect is due to the increase of adverse pressure gradient at the rear portion of the airfoil causing an increase in thickness of the boundary layer. The increase of the adverse pressure gradient is caused by the large increase of the maximum suction shown in the sections on Method of Small Perturbations and Applications of the Hodograph Method. Both effects can be taken into account—at least approximately—by the theoretical considerations given in the section on Method of Small Perturbations. It was pointed out that the velocity and pressure distribution along the surface of a slender body or an airfoil at a certain Mach number is approximately the same as in a flow of an incompressible fluid provided the body is expanded laterally in the ratio $1:\sqrt{1 - M_1^2}$. This reasoning furnishes a method for estimating the compressibility effect on the drag in the following way. Consider first, symmetrical sections. If the thickness ratio of the airfoil is denoted by t, the fictitious thickness corresponding to the Mach number M_1 is equal to $t/\sqrt{1 - M_1^2}$. Provided the influence of the thickness on the drag is known for the family of airfoils considered, the minimum drag coefficient C_{D_0} can be put approximately equal to the minimum drag coefficient C_{D_0} of a section of the thickness $t/\sqrt{1 - M_1^2}$. To estimate the correc-

tion for an arbitrary angle of attack it has to be taken into account that by the lateral expansion of the flow pattern the angle of attack is also increased in the ratio $1/\sqrt{1-M_1^2}$. Therefore, the drag coefficient of a section of thickness t at the angle of attack α and Mach number M_1 will be approximately equal to the drag coefficient of a section of the thickness $t/\sqrt{1-M_1^2}$ at the angle of attack $\alpha/\sqrt{1-M_1^2}$. If the effect of compressibility on the lift coefficient is known, one can use the drag coefficient of the expanded section at the lift coefficient C_{L_M}, which corresponds to the Mach number M_1 at the angle of attack α. For cambered sections the camber has to be also increased in the ratio $1:\sqrt{1-M_1^2}$.

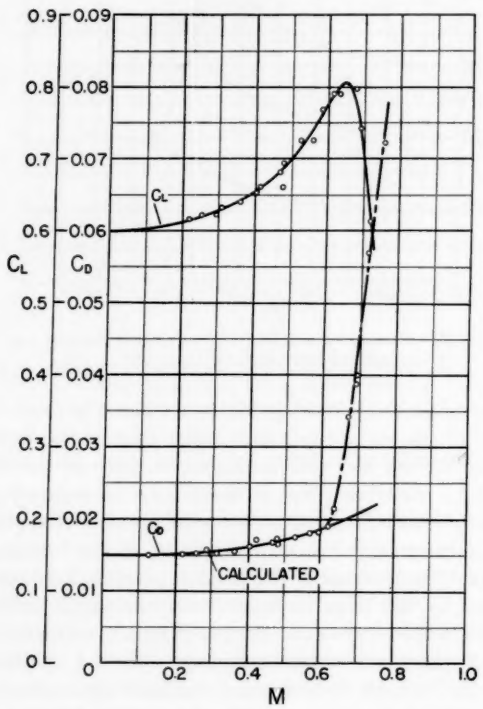

FIG. 14. Lift coefficient C_L and drag coefficient C_D for N.A.C.A. 4412 at $\alpha = 1°52'$.

Fig. 14 gives a comparison between values of the drag coefficient of the section N.A.C.A. 4412 measured at 1°52′ angle of attack and computed according to the method explained above. The influence of thickness was evaluated according to N.A.C.A. Technical Report No. 669 which apparently represents the last word in this somewhat controversial question. It was assumed that the relative influence of the thickness is independent of Reynolds number. It is seen that the accordance between the measured and estimated values is very good, until the "compressibility burble" occurs, indicated by simultaneous sudden drop of the lift coefficient and increase of the drag coefficient.

The "compressibility burble" is apparently caused by a shock wave extending from the surface to a certain distance from the surface. Since the magnitude of the pressure or the velocity jump is variable along the discontinuity surface, the flow behind the shock wave is a flow with continuously distributed vorticity even if the flow is irrotational in front of the discontinuity surface. Since far behind the airfoil as the flow becomes parallel again—at least theoretically—the shock wave must cause a non-uniform velocity distribution; *i.e.*, a wake in addition to the wake caused by the boundary layer. This has been experimentally shown by J. Stack, W. F. Lindsey, R. E. Littell and also by C. N. H. Lock.

There are only a few measurements available for drag and lift of airfoils beyond the velocity of sound and it seems that nobody has been able yet to obtain reliable measurements at the speed equal to the velocity of sound. The observations of the N.A.C.A. are extended essentially to the speed at which the compressibility burble occurs. The experimenters working with the high-speed tunnel in Guidonia found by increasing the speed beyond this limit a "zone of oscillation" in which no stable equilibrium can be established between the air forces and the balance. Beyond this zone the drag coefficient has a higher value and is apparently constant in the range of the velocity of sound. Fig. 15 shows the Italian measurements for an airfoil with 6.32 per cent thickness whose shape is almost identical with that of a Clark Y section. Curiously enough the N.A.C.A. measurements on a Clark Y airfoil of 6 per

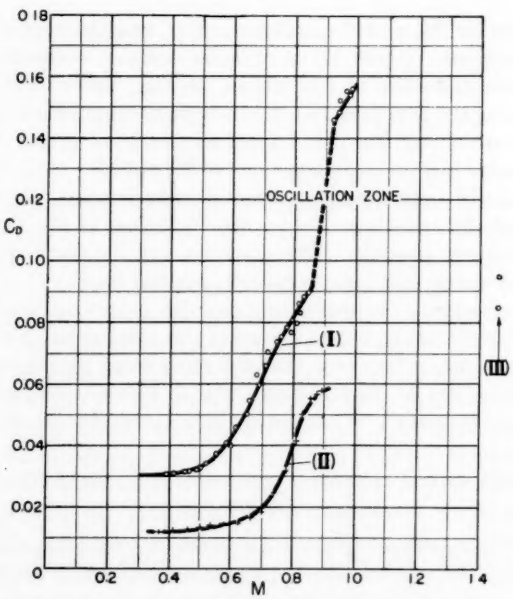

FIG. 15. Drag coefficient C_D *vs.* Mach number M, at $C_{L_0} \approx 0.45$.

(I) Italian propeller section of 6.32 per cent thickness (A. Ferri).
(II) Clark Y section of 6 per cent thickness (J. Stack).
(III) Göttingen 622 section of 8.85 per cent thickness for $\alpha = 2°$ and 4° at $M_1 = 1.47$ (A. Busemann and O. Walchner).

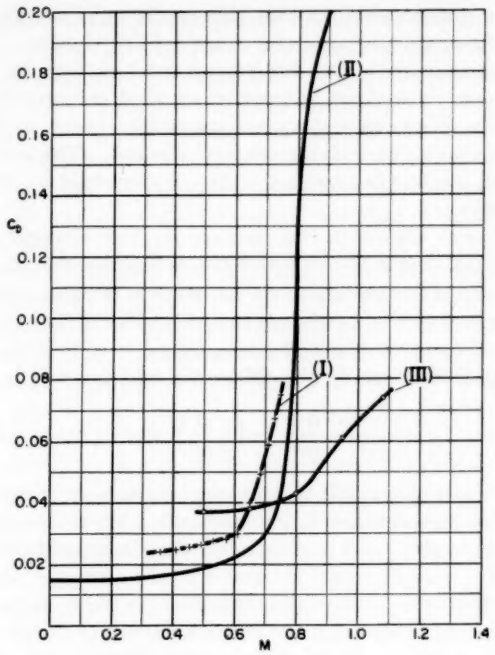

FIG. 16. Drag coefficient C_D of R.A.F. 6 section, 10 per cent thickness, at $\alpha = 2°$.
(I) N.A.C.A. tests (J. Stack).
(II) British propeller tests (G. P. Douglas).
(III) Open jet tests (L. J. Briggs and H. L. Dryden).

cent thickness are at wide variance with the Guidonia results. The two curves correspond to equal values of $C_L \approx 0.45$ for $M_1 \rightarrow 0$. For comparison the drag coefficients of a similar airfoil of 8.85 per cent thickness at $M_1 = 1.47$ for 2° and 4° angle of attack are indicated. These data are taken from measurements of A. Busemann. The N.A.C.A. data and results of British propeller tests concerning the compressibility effect on the drag of a 10 per cent thick R.A.F. 6 section at 2° angle of attack are in fair accordance (Fig. 16). In the same diagram the data obtained by Briggs and Dryden in an open high-speed jet are also plotted. It appears that these values are probably too low.

There are some measurements on lift and drag of a section with circular-arc upper and flat lower surface between $M_1 = 1.47$ and 2.13. These data are plotted in Figs. 17 and 18. In order to complete the picture as far as possible the corresponding data for the N.A.C.A. 4409–34 section are plotted for $0.35 < M_1 < 0.84$. This section is somewhat similar, having its maximum thickness at 40 per cent of the chord (instead of 50 per cent) and an almost flat bottom. It seems that the general shape of the drag curve is similar to that of the drag curve of a sharp-nosed projectile. The lift apparently decreases greatly after the first occurrence of the compressibility burble. The rise shown by the last points needs further confirmation. The theory of lift and drag of airfoils for supersonic speed has been developed by J. Ackeret, A. Busemann and S. G. Hooker.

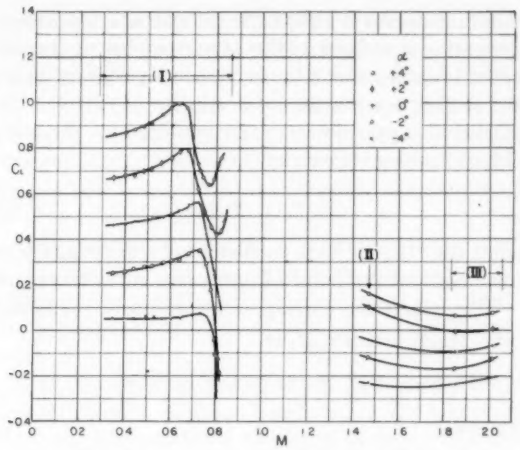

FIG. 17. Lift coefficient C_L *vs.* Mach number M.
(I) N.A.C.A. 4409–34 (J. Stack and A. E. v. Doenhoff).
(II) Circular-arc section of 8.85 per cent thickness (A. Busemann and O. Walchner).
(III) Circular-arc section of 8.8 per cent thickness (A. Ferri).

There are no data published on the compressibility effect on the drag of streamlined bodies such as fuselages and nacelles. In general it can be expected that if a certain streamlined shape has a favorable drag coefficient at low speed, it has to be expanded in the length direction or contracted laterally in a linear ratio about equal to $1:\sqrt{1 - M_1^2}$ in order to keep the drag coefficient low at high speed.

THE CRITICAL MACH NUMBER

The Mach number at which the compressibility

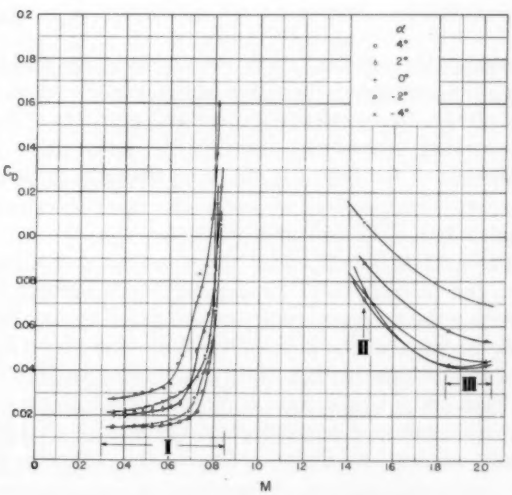

FIG. 18. Drag coefficient C_D *vs.* Mach number M.
(I) N.A.C.A. 4409–34 (J. Stack and A. E. v. Doenhoff).
(II) Circular-arc section of 8.85 per cent thickness (A. Busemann and O. Walchner).
(III) Circular-arc section of 8.8 per cent thickness (A. Ferri).

burble first occurs is called the critical Mach number. In general it is defined as the Mach number of the undisturbed flow for which the local velocity at some point of the surface reaches the local velocity of sound. Using Glauert's approximation the condition for the critical Mach number is given by

$$V + v_x' + a_1 \tag{66}$$

Assume that the pressure coefficient of a given airfoil or slender body at the point of maximum suction is equal to $-C_{p_0}$; then the corresponding pressure coefficient for the Mach number M_1 is

$$-C_{p_M} = -C_{p_0}/\sqrt{1 - M_1^2}$$

and since

$$C_{p_M} = 2v_x'/V$$

the Eq. (66) can be written in the form

$$V[1 + (C_{p_M}/2)] = a_1$$

or

$$M_1\left[1 + \left(C_{p_0}/2\sqrt{1 - M_1^2}\right)\right] = 1 \tag{67}$$

Hence, one obtains the following equation for the critical Mach number

$$2[(1 - M_1)^{1/2}(1 + M_1)^{1/2}]/M_1 = C_{p_0} \tag{68}$$

E. Jacobs, by using Eq. (8) for the relation between pressure and velocity, obtained the expression:

$$C_{p_0} = 2\sqrt{1 - M_1^2}\left[1 - \left(\frac{2 + (\gamma - 1)M_1^2}{\gamma + 1}\right)^{\gamma/(\gamma-1)}\right]\Big/\gamma M_1^2 \tag{69}$$

However, the question arises whether it is justified to introduce expressions involving higher terms in a linearized theory that neglects squares and products of the perturbation velocities. Possibly the error made in evaluation of the ratio C_{p_M}/C_{p_0} is of the same order as the correction involved by use of the exact thermodynamic relation (8) instead of (66).

The method developed by Tsien and the author permits the calculation of the critical Mach number by equating the pressure coefficient computed by Eq. (62) with the value of the pressure coefficient corresponding to a local Mach number $M = 1$ according to Eq. (8). In Figs. 10 to 12 this limit is plotted by a full line.

Fig. 19 shows a comparison of the relation "critical Mach number" *vs.* C_{p_0} according to the different approximate theories. For comparison a few experimental results are also indicated.

Of course, by laborious analytical methods it would be possible to refine the calculation of the critical Mach number, defined by the first coincidence between local velocity and local velocity of sound. However, it appears questionable whether this is the right criterion for the first appearance of a compressibility burble. A closer examination of the fundamental mathematical problem shows that at least in an ideal compressible fluid a shock wave does not necessarily occur if the local velocity exceeds the velocity of sound. The hodograph method can be used to throw some light on this question.

It was shown in the section on the Hodograph Method, that if $\psi(v, \Theta)$ is the stream function for a flow pattern of a compressible fluid where v is the magnitude of the velocity and Θ the angle of inclination of the velocity vector at an arbitrary point, then ψ as a function of v and Θ must satisfy the differential equation

$$\frac{1 - M^2}{\rho^2}\frac{\partial^2\psi}{\partial\Theta^2} + \left(\frac{v}{\rho}\frac{\partial}{\partial v}\right)^2\psi = 0 \tag{70}$$

Now it is possible to find exact particular integrals of this differential equation, and such solutions are supposed to give exact flow patterns of a compressible ideal fluid. The physicist or engineer is, of course, not satisfied by having determined streamlines in the hodograph plane and is anxious to see the flow pattern in the physical x, y plane. If the function $\psi(v, \Theta)$ is known, the computation of streamlines $\psi(x, y) = \text{const.}$ in the physical plane requires a further integration. However, in performing this integration it is found that in some cases a streamline that is perfectly continuous in the hodograph plane shows a singular behavior: plotting successive points of a streamline in the physical plane at a certain point which can be called the critical point, the streamline cannot be continued in the flow direction, and gives a quite impossible flow pattern with two families of crossing streamlines in the same plane. The velocity at this critical point is neither zero nor infinite. However, by continuing the calculation one obtains a second branch of the streamline, which meets with the first branch in a kind of "cusp." It appears that the flow cannot enter in a certain "forbidden"

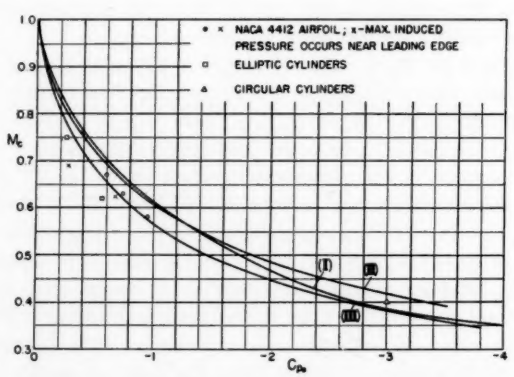

FIG. 19. Critical Mach number M_c *vs.* maximum suction-pressure coefficient at low speed C_{p_0}.
(I) Eq. (68).
(II) Eq. (69) (E. Jacobs).
(III) Hodograph method (Th. von Kármán and H. S. Tsien).

region without violating one of the physical assumptions underlying the analysis. The two assumptions are: continuity and irrotationality. Since the continuity cannot be violated, it must be assumed that the flow becomes rotational. In an ideal compressible fluid this can happen only through a pressure jump; *i.e.*, a shock wave. In this sense the occurrence of such critical points indicates the necessary appearance of shock waves.

The phenomenon of critical points was found first by M. and F. Clauser who, on the suggestion of the present author, calculated a family of exact solutions of the Eq. (70) and discussed the corresponding flow patterns. They investigated especially the flow bounded by two inclined straight walls. Recently, F. Ringleb, by a method similar to that employed by the Clausers, found exact flow patterns of compressible fluids. He presents in his paper one example which is especially instructive as to the question of the first appearance of shock waves and will be briefly discussed here.

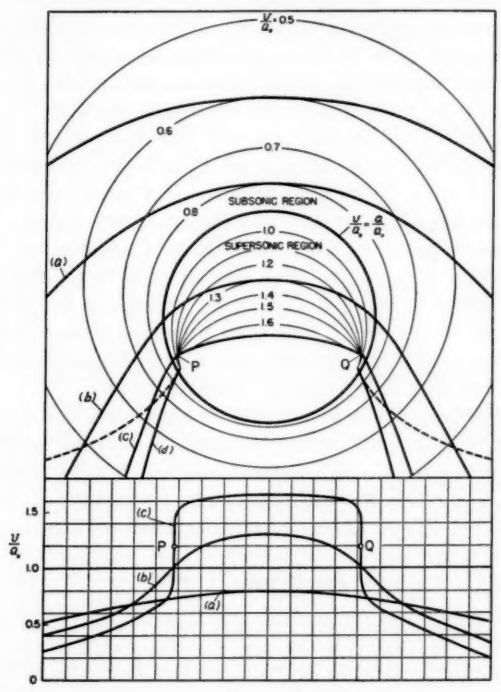

FIG. 20. Flow in a curved convergent-divergent nozzle (F. Ringleb).

Fig. 20 shows a kind of nozzle which is essentially a curved convergent-divergent nozzle. The boundaries are exact streamlines of a compressible fluid obtained in the manner explained above. The velocity is zero at the entrance and at the exit at infinity. The lines of constant velocity—which happen to be exact circles—show that the flow is partly subsonic, partly supersonic. Consider first the nozzle bounded by the lines (*a*) and (*b*). The velocity distribution along the two walls is given by the lines (*a*) and (*b*) in the lower part of Fig. 20. It is seen that the flow is perfectly smooth and continuous. However, if the nozzle is bounded by the lines (*a*) and (*c*) the velocity has infinite slopes at the points P and Q. If one plots another streamline proceeding forward inside, it is found that the streamline stops at a certain point and turns back, crossing the nozzle; this is, of course, physically impossible.

It is remarkable that the maximum velocity reached at the wall (*c*); *i.e.*, in the limiting case, is as high as $1.67a_0$ (where a_0 is the velocity of sound at rest) and that the velocity at the point where a discontinuity first appears in the flow is equal to $1.20a_0$.

The German author does not discuss further the conditions for the singular behavior of the streamlines. However, the equations developed in the section on the Hodograph Method of this paper make it possible to determine a condition for the occurrence of an infinite acceleration. Obviously the acceleration in the direction of the flow is equal to $(\partial v/\partial s)(ds/dt) = (\partial v/\partial s)v = (\partial v/\partial \varphi)v^2$. Hence, if v has a finite value, the acceleration can be infinite only if $\partial v/\partial \varphi = \infty$. Now according to Eq. (48)

$$\partial v/\partial \varphi = -(1/\Delta)(\partial \psi/\partial \theta)$$

where $\Delta = (\partial \varphi/\partial v)(\partial \psi/\partial \theta) - (\partial \varphi/\partial \theta)(\partial \psi/\partial v)$ is the so-called Jacobian previously discussed. If the case $\partial \psi/\partial \theta = \infty$ is excluded for the time being, $\partial v/\partial \varphi$ can be infinite only when

$$\Delta = (\partial \varphi/\partial v)(\partial \psi/\partial \theta) - (\partial \varphi/\partial \theta)(\partial \psi/\partial v) = 0 \quad (71)$$

Substituting the values of $\partial \varphi/\partial v$ and $\partial \varphi/\partial \theta$ from the Eq. (50) one obtains the condition

$$(1 - M^2)(\partial \psi/\partial \theta)^2 + v^2(\partial \psi/\partial v)^2 = 0 \quad (72)$$

The expression on the left side is obviously always positive if $M < 1$. Therefore, the condition (72) can be only satisfied if $M > 1$; *i.e.*, if the flow is supersonic. Eq. (72) has a simple geometrical meaning. Denote the angle of inclination between a circle $v = \text{const.}$ and the tangent to the line $\psi = \text{const.}$ at an arbitrary point of the hodograph plane by β. Then

$$\tan \beta = -\left(\frac{1}{v}\frac{\partial \psi}{\partial \theta}\right)\Big/\frac{\partial \psi}{\partial v} \quad (73)$$

Hence, from Eq. (72) follows

$$\tan^2 \beta = 1/(1 - M^2) \quad (74)$$

In the theory of supersonic flow the angle $\sin^{-1}(1/M) = \alpha$ is called Mach's angle; it is equal to the angle between the velocity vector and the so-called stationary wave-front produced by a small perturbation applied to the fluid. Hence, according to Eq. (74), $\beta = \pm \alpha$. The lines intersecting the circles $v = \text{const.}$ at the angle α are called the *real characteristics* of the differential Eq. (70) and for given constant γ are epicycloids. Hence, infinite acceleration can occur at points where the curve $\psi = \text{const.}$ in the hodograph plane is tangen-

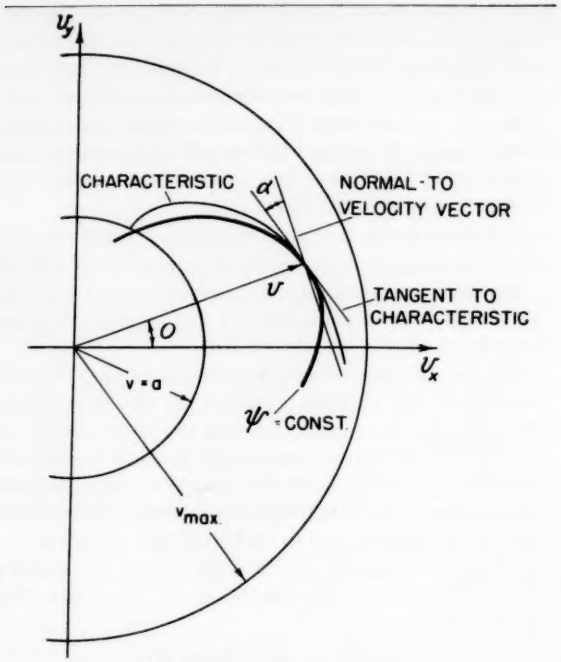

FIG. 21. Condition of infinite acceleration and breakdown of isentropic irrotational flow.

tial to a characteristic curve of the differential Eq. (70)* (Fig. 21).

The direction of the tangent to a ψ = const. curve indicates the direction of the infinitesimal change of the velocity vector; *i.e.*, the direction of the resultant acceleration, in the sense that the angle between the velocity and acceleration vectors is the same in the hodograph and physical planes. Hence, at the point where infinite acceleration occurs the angle between the resultant acceleration and the normal to the velocity is equal to the Mach angle. Applying this result to the case that the velocity along a streamline changes from subsonic to supersonic and back to subsonic again one can show that in order to avoid infinite acceleration the angle between the acceleration and the normal must be smaller than the local Mach angle in the entire supersonic range. Further investigations are necessary to expand the physical meaning and importance, if any, of this finding.

In Fig. 22 streamlines for the flow pattern shown in Fig. 20 are plotted in the hodograph plane. Fig. 23 refers to the velocity distribution found experimentally for the N.A.C.A. 4412 section at various Mach numbers. The lines are plots of the "zero streamline"

* The fact that the singular behavior of the streamlines occurs at points where streamline and characteristics in the hodograph plane have the same direction was first found by the Clausers. The condition (71) is mentioned in the paper of Ringleb. It appears that if a streamline is tangent to a characteristic in the hodograph plane, in the physical plane the characteristics (wavelets) of the same family meet; their envelop probably introduces the first discontinuous wave front.

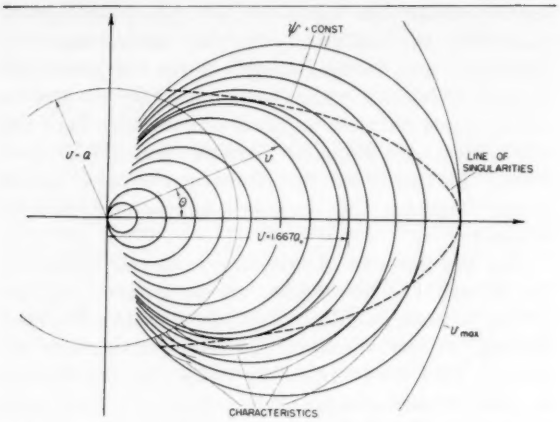

FIG. 22. Streamlines and characteristics in hodograph plane for a curved convergent-divergent nozzle.

(upper boundary) in the hodograph plane; in other words the velocity vectors occurring along the boundary are plotted as to magnitude and direction and their end points connected by a curve. It is seen that the line at a point close to the point where the shock wave occurs is tangential to characteristics passing through the same point. However, this coincidence might be only accidental because the velocity distribution is greatly changed by the shock wave, whereas the theorem proved above refers to the first occurrence of an infinite slope of the pressure, but not to a finite pressure jump.

Flow patterns with continuous transition of the velocity of sound in both directions have been calculated before by G. I. Taylor, W. Tollmien and recently by H. Görtler who investigated the flow along a wave-shaped solid wall by computing a second approximation on the lines of the Glauert-Prandtl method.

The author does not wish to exaggerate the importance of the above considerations. It is merely shown that in the case of an ideal compressible fluid continuous, shockless flow patterns can be constructed in which the velocity of flow reaches magnitudes as great as 1.6

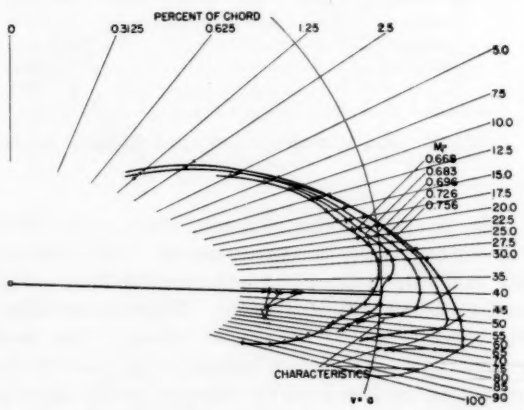

FIG. 23. Representation of upper surface of N.A.C.A. 4412 in hodograph plane at $\alpha = -2°$.

times the velocity of sound and becomes subsonic again. However, this does not prove yet that such flow patterns are really possible in a real fluid. Furthermore, it is possible that the flow with shock wave is more stable than the continuous flow pattern. If a body moves with supersonic velocity in a compressible fluid, shock waves appear unavoidable, *i.e.*, flying with supersonic velocity will always involve wave resistance in addition to the frictional drag. However, it appears that careful theoretical and experimental research might be able to push the velocity of flying nearer to the velocity of sound than is possible now. The mere fact that the air passes over a wing with supersonic velocity does not necessarily involve the occurrence of a compressibility burble and energy loss by shock wave.

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Letter to the Editor

June 12, 1941

Dear Sir:

The following discussion applies to the article, *Permanent Buckling Stress of Thin-Sheet Panels under Compression*, by Messrs. W. L. Howland and P. E. Sandorff published in the May, 1941, issue of the Journal.

On page 264, the authors derive a value for the strain at the defined yield strength (Equation (11)) and subsequently use this strain as the maximum allowable total strain for the panel. The value which the authors use is apparently based on tensile-test data for 24ST Alclad sheet. Long before this strain is reached, the Alclad coating begins to yield. (Yield strength of commercially pure aluminum is 4000 lbs. per sq.in.; see *Structural Aluminum*, Aluminum Company of America, 1938, page 10.) In a tensile (or compression) test, the core material, because of its greater thickness, has a predominating influence. Therefore, if the core material has not yielded, the residual strain in the Alclad sheet will be small. When initial bending of the sheet is involved, as in buckling, the yielding of the Alclad coating, which is a large distance from the neutral surface, is more important than in the case of the tensile-test, and when permanent bending is involved, as in the present case, the yielding of the Alclad coating has a tremendous effect. (This has recently been checked by tests at the research laboratories of the Aluminum Company of America.) It is therefore believed that the value of maximum strain based on tensile test data has little significance in connection with permanent buckling of Alclad sheet, which depends wholly on the bending characteristics of the sheet. Since the authors subsequently (page 265) chose an empirical value for the maximum strain, the preceding comments on the action of Alclad sheet are not too important in connection with the value used for the maximum strain. However, the fact that the Alclad covering yields at low strains is of utmost importance in interpreting the buckle-depth data.

Referring to Fig. 5 on page 265, when the specimen was first stressed to 5500 lbs. per sq.in., the total strain was great enough to cause yielding in the Alclad coating. When the load was released, the bending strength of the unyielded core was sufficient to return the sheet practically to its original shape despite the slightly yielded Alclad coating. However, when the critical stress was approached in the next loading, the effect of the prestressed Alclad coating, being magnified by the presence of the compressive stress, appreciably increased the buckle depth. Only when the stiffener stress reached the previous maximum of 5500 lbs. per sq.in. was the buckle depth the same as the buckle depth during the first loading. After continuing the loading up to 7800 lbs. per sq.in. and then releasing, the core again returned the sheet practically to its original shape. On the next loading, the Alclad coating, having more residual stress than on the previous loading, caused still deeper buckles in the vicinity of the critical stress (below previous stiffener stress). On subsequent loadings, to higher stresses, the increased yielding of the Alclad coating was sufficient to cause measurable permanent buckles at the first increment of load. With this explanation of the data plotted on Fig. 5, the authors' conclusion (see page 268), "That an ideal (absolutely flat) panel once subjected to stress greater than the critical-buckling stress will exhibit some residual roughness at any lower reference stress, even at no load," is not "difficult to believe."

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Vought-Sikorsky Aircraft Division,
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Fog Study of Miami, Florida

JOHN RANDLE DeHART

Eastern Air Lines, Inc.

As advection plays an important role in every case of low visibility for this area, the geographical location of Miami is of great importance. Miami has an extreme water exposure, located as it is, with the Atlantic Ocean on the east and with the Gulf of Mexico approximately 75 miles to the west. To the northwest is Lake Okeechobee and the adjacent Everglades.

The Everglades contribute not only moisture to the lower layers of air, but also at frequent intervals add quantities of smoke from burning muck. This may or may not be a contributing factor to fog formation, but certainly it does occasionally cause visibilities to become so low as to be dangerous to aircraft. In fact, one-third of Miami's visibilities of one mile or less, at surface winds of less than 5 miles per hour, are due to smoke.

In view of the topography, almost any air arriving at Miami, regardless of its path, must have had some recent water history. The graphic story of this is told in Fig. 1, where it can be seen that air, arriving over trajectory *A*, has its path almost entirely over the Everglades,

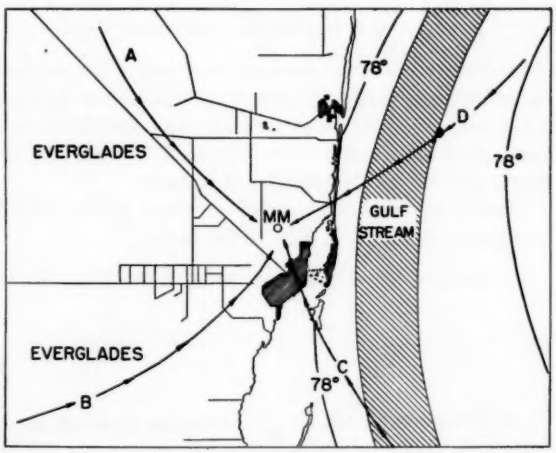

FIG. 1. Topography and trajectories.

while air arriving over trajectory *B* has had some history over the Gulf of Mexico, and only the latter part of its journey was over the Everglades. Air in both trajectories *A* and *B* will become stabilized during the night when the air temperature is higher than the temperature of the water of the swamp land.

Awarded first prize for 1940 by the Air Transport Association of America "for the best research, by an airline employee, having practicable application to meteorology or dispatching."

This is the first such award given by the Air Transport Association of America.

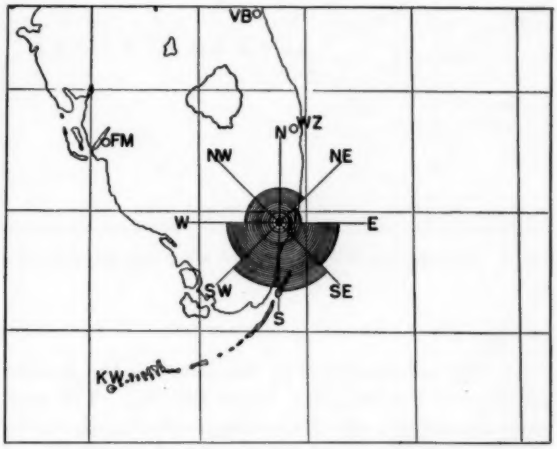

FIG. 2. Distribution of fog by gradient direction.

Air in trajectory *C*, which has a high moisture content due to its water history over the Caribbean, is stabilized by its passage over the progressively cooler shore waters. Later when it passes over the city, it is exposed to smoke and dust pollution. These two factors tend to cause the largest amount of fog of any other gradient direction.

In consideration of trajectory *D:* This air is inherently unstable because of its passage over the warm waters of the Gulf Stream, the edge of which is approximately 12 miles from the airport. In this short path the air has little opportunity to cool and become stable, except with the very lightest of winds.

A gradient windrose shows air arriving from NE to E produces the least amount of fog of any other gradient direction, and that from E to SW produces the greatest amount; the first because of this unstabilizing effect of the Gulf Stream, and the second because its path has been parallel to the sea isotherms of the Gulf Stream and during the latter part of its history it has become stabilized over the cooler bay waters. (See Fig. 2.) Fog occurs with winds from SW to W more often than from W to NE because this air is usually more moist due to its water history over the Gulf of Mexico.

Compare this gradient windrose with the surface windrose of restricted visibility as given in Fig. 3*a*, which shows a maximum of low visibility from NW. However, this surface chart includes in it low visibility due to smoke and rain, while the gradient windrose excludes both of these. The light gradients from SE to S would most likely show up as calms on the surface windrose.

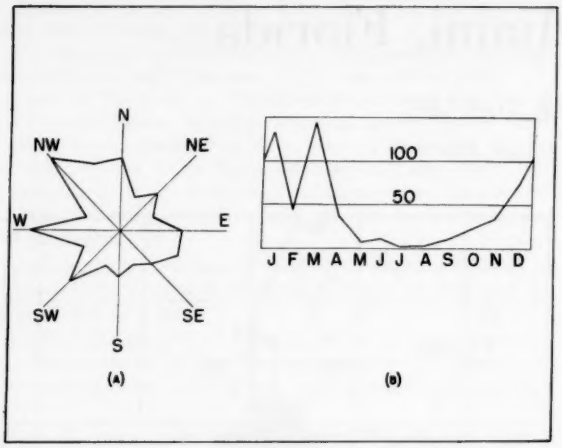

FIG. 3. Distribution of fog by surface winds and distribution of fog by months.

SEASONAL VARIATIONS

Another consideration in this study is the seasonal variation of low visibility. (See Fig. 3*b*.) The maximum amount of fog in this area occurs during the month of March, which only slightly exceeds the fog formation during January. The drop in the fog curve for February may be explained in that this month has the highest average wind velocity, and the fact that low visibilities occur less frequently from May through September is due to the greater amount of insolation at this time.

In breaking the cases of fog down into their respective air masses the following result was found, which is about the normal occurrence of these air masses:

Tropical Maritime	43.5 per cent
Modified Polar Continental	38.7 per cent
Modified Polar Pacific	17.8 per cent

Therefore, it can be seen that fog may form in any air mass that covers Miami. Stratus occurs very rarely at Miami, if at all, unless in conjunction with a front. These frontal cases have been discussed fully in such papers as George's *The Causes and Forecasting of Low Ceilings and Fogs at Jacksonville Airport* (published by Eastern Air Lines, Inc., 1939), and will not be restated here.

QUALITATIVE ESTIMATION

Whether or not fog will form, and if so, will it be light or dense, may be computed from the curves shown in Fig. 4, which are given for both Tropical and Polar Air Masses. These curves are not to be used if local showers have occurred in the four hours preceding the 1930 E map because the dew point would then be fallaciously high. To use these curves, plot gradient wind as given by isobars on the 1930 E map against the dew point depression as given on that same map. If the point so plotted falls between the inner and outer curves, fog will occur, but the visibility will not be below one-half mile; but if it lies within the inner circle the fog which occurs will cause the visibility to be between zero and one-half mile.

The gradient wind must also be forecast for the early morning hours because a point that plotted in the light fog sector might move into the dense fog sector if the gradient wind should decrease; the reverse might be true if the gradient wind should increase.

TIME ESTIMATION

After deciding that fog will form, the next step is to determine the time of fog formation. Using the curve of Fig. 5, plot the sunset dew point depression on the curve and read the time of formation from the abscissa. The curve assumes constant gradient winds between 9 to 16 m.p.h. For decreasing winds fog forms 1 to 3 hours earlier than indicated; and for increasing winds, 1 to 3 hours later. This curve, of course, is to be used when the forecaster expects the cloud cover to dissipate soon after sunset. If this is not the case, suitable corrections must be made.

To correct for the presence of cloud cover, the forecaster must add 75 per cent of the number of hours that the cloud cover is expected to remain, to the value derived from the time of formation curve.

The curves themselves are empirical and were derived from plotting U.S. Weather Bureau data from Eastern Air Lines weather charts.

FORECASTING RADIATION-ADVECTION FOG

Presentation of the following two cases, (1) a normal radiation-advection fog, and (2) a case when fog did not form, show how the forecast was simplified and more accurate results were obtained by the use of charts and curves compiled for this study.

Ground fog may be expected in Miami in the normal case shown in Fig. 6, in view of the facts:

1. Isobars show gradient wind from 110° at 15 m.p.h.

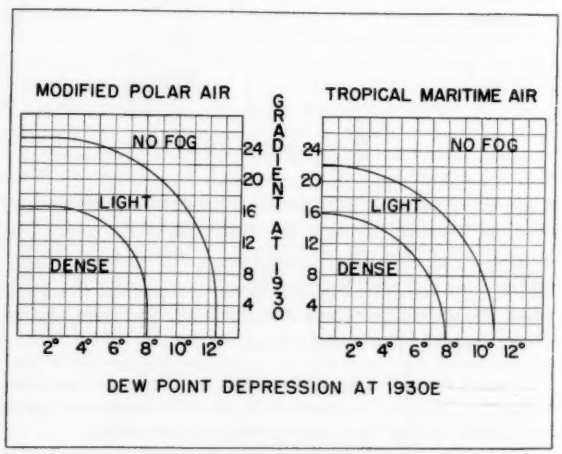

FIG. 4. Qualitative forecasting curves.

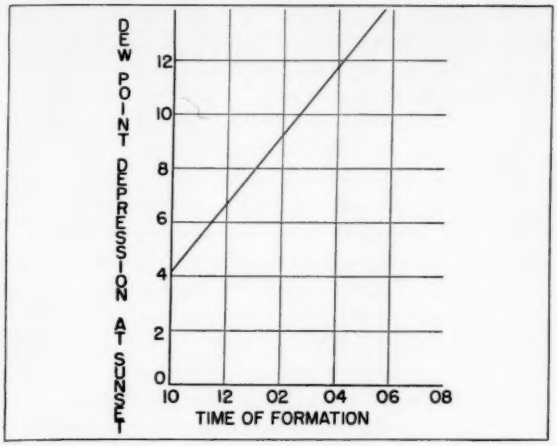

FIG. 5. Time of formation curves.

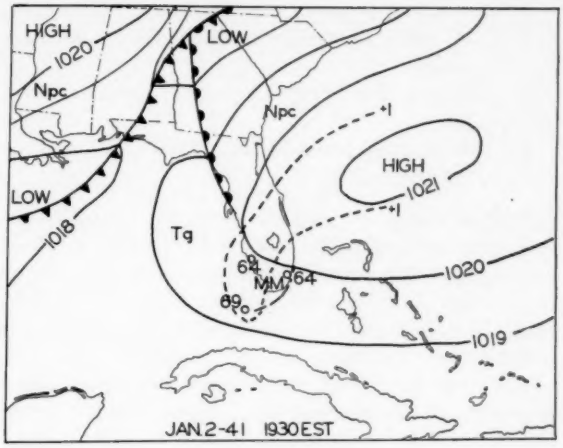

FIG. 6. Normal case.

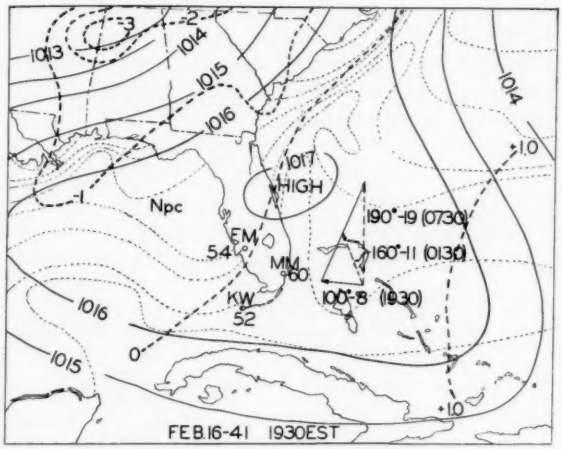

FIG. 7. Case when fog did not occur.

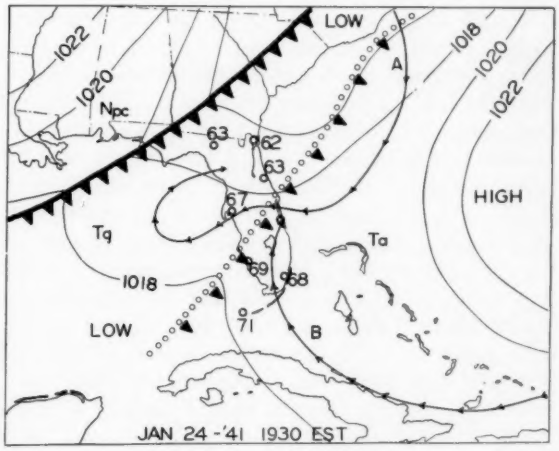

FIG. 8. Formation of *Ta Tg* front.

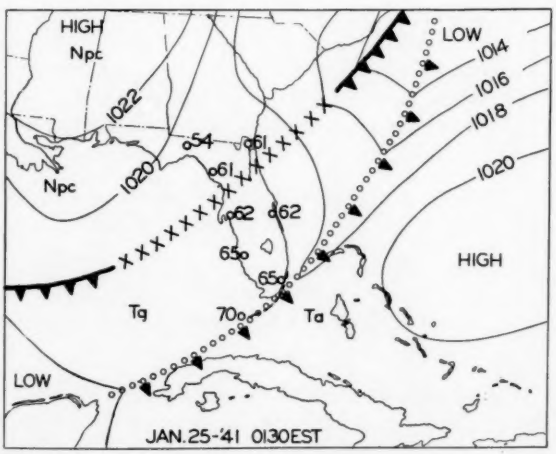

FIG. 9. Passing of front.

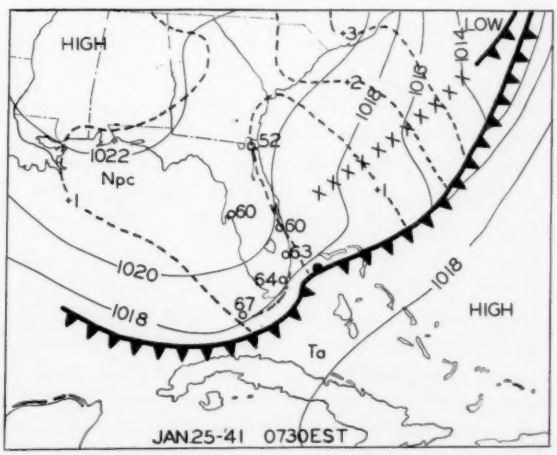

FIG. 10. Lifting and breaking of fog.

2. Decreasing gradient winds are expected because of rising tendencies at Key West and Fort Meyers.
3. Gradient of moisture shows no advection of drier air.
4. Dew point at sunset in Miami was 64°F., and the temperature 74°F.

With a dew point as high as 64°F., Miami is undoubtedly in Tropical Maritime Air. Plotting gradient wind against dew point depression at 1930 E on Fig. 4 for Tropical Maritime Air, the point falls in the light area. But, because the gradient wind is decreasing, the point will move from the light to the dense fog area.

The result of plotting the temperature and dew point depression on the curve of Fig. 5 shows that fog should form at 0200 E. Actually, at 0135 E visibility was reduced to one-half mile. By 0235 E visibility was reduced to one-fifth mile in dense ground fog, and remained between one-eighth and one-half mile until 0700 E.

The importance of forecasting and of noting all details such as gradient of moisture, forecasting gradient wind and trajectory cannot be overlooked, because the history of the air arriving at Miami can be derived only from the few stations that are located on the Florida peninsula. As, for example, in Fig. 7, using the gradient wind from the map, with a temperature of 67°F. and a dew point of 60°F., without further investigation it would appear that dense ground fog would form. However, employing the graphical method of forecasting the gradient wind, using an isallobaric component as shown on Fig. 7, it is clear that the gradient wind should be 160° at 11 m.p.h. by 0130 E, and 190° at 19 m.p.h. by 0730 E. From this increasing gradient it is apparent that any fog forming would be light. Further study shows dew points of 52°F. at Key West and 54°F. at Fort Meyers; therefore, any advection of air at Miami would be drier air, further reducing the possibility of dense fog.

Actually, the balloon runs gave these values:

1800 E	110°	5 m.p.h.
2500 E	180°	10 m.p.h.
0600 E	230°	16 m.p.h.

and the visibility did not go below two miles at any time during the night. The following morning showed that the dew point had dropped to 51°F. Such accurate computations of wind acceleration are, of course, possible only with unusually smooth isallobars.

Forecasting Post Cold Frontal Fog

Another type of fog peculiar to this area is the post cold frontal fog brought about by two tropical air masses. The following sequence of events, of which this is a typical example, produces the worst cases of fog and low ceilings at Miami. (See Fig. 8.) The *Tg* air, shown as trajectory *A*, has had an approximate three-day history over water. It moves eastward in the circulation brought about by the advancing cold front shown in the upper left-hand corner of the map. The *Tg* air underruns the very warm *Ta* air, and a NE–SW line of frontogenesis is formed. This frontogenetic line continued southward and crossed Miami near 2300 E.

Fig. 9 shows the situation 6 hours later. The line of frontogenesis has now become a true *Ta*-*Tg* front, and the original cold front is undergoing frontolysis. At Miami the stabilizing effect of the warm air aloft, plus the rapid clearing behind the front coupled with low wind velocities, and perhaps some mixing, caused visibility to be reduced to one-half mile by 0335 E, and to one-eighth mile by 0635 E. This fog lifted to stratus due to the advection of unstable air brought about by a wave formation off the coast as shown in Fig. 10. This same condition lowered ceilings to zero and visibilities to one-eighth mile at Key West, a most unusual occurrence.

The author feels that careful consideration and weighing of all available data should result in the reasonably accurate fog forecasting in at least 80 per cent of fog cases, and that errors in forecasting are directly attributable to the lack of sufficient time for proper consideration, to inadequate data, or to carelessness on the part of the forecaster.

Conclusions

Combination of conditions under which fog will form and visibility will lower to one-half mile or below:

1. Forecast from 1930 E map.
2. Showers have occurred two to four hours previous.
3. Gradient wind will not accelerate to 20 m.p.h. by 0730 E.
4. Dew point depression on 1930 E map is not more than 11°F.

or

1. Forecast from 1930 E map.
2. Showers have not occurred two to four hours previous.
3. Gradient wind is below 10 m.p.h. and shows deceleration, or at least no acceleration.
4. Dew point depression on 1930 E map is not more than 5°F.

Combination of conditions under which fog will form and visibility will lower to one-half to two miles:

1. Forecast from 1930 E map.
2. Dew point depression is between 5°F. and 11°F.
3. Gradient wind not less than 10 and not more than 20 m.p.h. up to 0730 E.

This study was made from Eastern Air Lines' maps, drawn and analyzed at six-hour intervals from 1935 to 1940. All frontal analyses and air mass designations are their own.

The Use of Liquid Oxygen for High-Altitude Flying

JOHN D. AKERMAN
University of Minnesota

SUMMARY

A method and apparatus is described to produce gaseous oxygen for use in high-altitude flying, from liquid oxygen. The elimination of the danger of discontinuity of oxygen boiling and provision for securing the required volume of oxygen gas is accomplished by keeping the liquid oxygen under a predetermined pressure and supplementing the normal evaporation by spilling a necessary amount of liquid oxygen in the gas contained. Fast evaporation takes place to maintain a constant supply of oxygen gas inside the apparatus which is connected with a second-stage reducing valve of a B-L-B oxygen mask as developed by the Mayo Foundation.

To secure the spilling of liquid oxygen into the gas container, an absolute pressure valve is provided in the gas contained to keep a predetermined pressure of gas in the liquid container. The liquid container does not require the use of a vacuum bottle. Although no attempt is made to design the apparatus for any specific airplane, the mechanism is flexible for adaptation to meet any specific requirements of oxygen supply. Apparatus for handling and storing liquid oxygen and the possibilities of a portable liquid oxygen producing plant are discussed.

INTRODUCTION

LIQUID OXYGEN as a source of supply of oxygen gas for high-altitude flying was used extensively by the Military Service from 1925 to 1934. Since that time, for convenience and safety, the use of compressed oxygen gas has been introduced. Why the use of liquid oxygen was abandoned is not a matter for discussion in this paper, but there is sufficient evidence that with improper liquid oxygen installations there is a danger of shortage of oxygen gas during flight when descending from higher to lower levels. This fact was brought out by Dr. Jean Piccard* whose conclusions were:

During ascent the boiling point of oxygen goes down due to decrease of pressure. On descent, due to increase of pressure, oxygen stops boiling. At this period the pilot will be short of oxygen. There are several cases on record, here and abroad, where such a situation was encountered by pilots using liquid oxygen installations.

After discussing with the author the possibilities of forced vaporization of oxygen under controlled pressure, Dr. Piccard fully endorsed this system of using liquid oxygen as supply for high-altitude flying. No doubt, the problem would not have reached its present stage without his help.

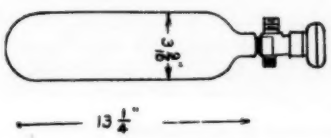

FIG. 1. Small size oxygen gas cylinder capacity (1 liter of liquid oxygen = approximately 880 liters of gaseous oxygen).

Fig. 1 shows a compressed oxygen gas cylinder sufficient for one man for 1 hour and 59 minutes at 30,000 ft. altitude if permissible pressure is 1800 lbs., but if the pressure is reduced to 400 lbs. it is sufficient for only 26 minutes at 30,000 ft. The decrease of pressure from 1800 lbs. to 400 lbs. decreases the contents from 203 liters to 45 liters. One liter of liquid oxygen would produce approximately 880 liters of gaseous oxygen at sea level and would be sufficient for one man for approximately $8^1/_2$ hrs. at 30,000 ft. Installation with 1800 lbs. pressure was satisfactory until it was discovered that if such a cylinder is hit by a bullet, an explosion occurs which is very damaging to the aircraft.

Therefore, specifications are now in force to make new installations of compressed oxygen on military aircraft in containers with a maximum pressure as low as 400 lbs. per sq.in. Such a container for the same amount of oxygen as shown in Fig. 1 would be 5.1 times larger and presents considerable problems of installation on modern aircraft. There is also the problem of providing armor against machine gun bullets.

To introduce again the use of liquid oxygen in order to save weight and space, and to meet requirements for large volume of oxygen, seems very promising except for two major difficulties. The first and easiest is technical development, and the second, and hardest, prejudice created by past experience.

This paper will be devoted to the first item, leaving the combat of prejudice to the merits of the development. Before going into details of the method and equipment, it is essential to make clear the basic conditions governing this work.

Presented at the Physiological Problems session, Ninth Annual Meeting, I.Ae.S., New York, January 29, 1941.

Charles Cole, Research Assistant, University of Minnesota Engineering Experiment Station, Minneapolis, Minnesota, and Norvin Erickson, Research Assistant, Mayo Foundation, Rochester, Minnesota, aided in the construction of apparatus and in the collection of data.

* *Change of Boiling Point of Liquid Oxygen at High Altitudes*, Dr. Jean Piccard.

1. No attempt is made to produce an actual installation for any airplane, commercial or military, because requirements and conditions vary so that it is impossible for educational institutions with limited financial means, to meet any conditions specifically. Specific adaptation and expensive refinements can be left for the future.
2. Requirements for the amount of oxygen for one man at different altitudes used in this work are those supplied by the Aviation Medicine Division of the Mayo Foundation.
3. Equipment is designed to be usable directly with the B-L-B mask developed by the Mayo Foundation.
4 Accepted as standards are flow meters as used by the Mayo Foundation since some other equipment has shown dangerous variations at high altitudes.
5. Difficulties of producing, handling and storing liquid oxygen may be in the future overcome with technical developments described later in this paper.

The first requirement in the use of liquid oxygen is the elimination of the possibility that there will not be enough oxygen gas delivered at some time during the flight.

The second requirement is the elimination of the possibility of building up high pressure inside of the container which might then burst and release oxygen gas at high pressure.

There is an unfounded fear that spilled liquid oxygen or oxygen gas at low pressure is a dangerous explosive by itself.

Fig. 2 shows a diagram of the setup in connection with the B-L-B mask with standard second-stage reducing valve, to which is connected a supply hose directly from the liquid oxygen supply at the same pressure (min. 45 lbs. to max. 80 lbs. per sq.in. absolute) as it would be from a compressed oxygen cylinder after going through the first-stage reducing valve on the B-L-B installations.

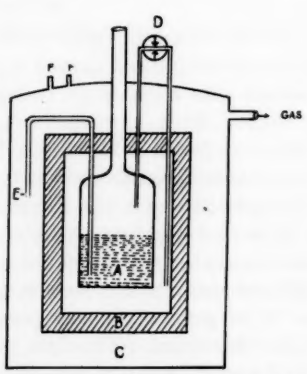

FIG. 2. Sketch illustrating principle of forced evaporation.

Liquid oxygen (a) is in a galvanized sheet-iron container. A vacuum bottle can be used if the cost is justified, but because of the low temperature of the surrounding, perforated insulation container filled with glass wool (b) it is not a necessity. A similar insulating effect is produced by allowing cold gas to seep through the space (b) filled with glass wool.

As the pressure rises simultaneously in the containers (a), (b), and (c), 45 lbs. absolute is reached immediately and the supply of gas is ready for use. If at any time, the pressure in (a), (b) and (c) goes above 80 lbs. absolute, or any other predetermined pressure, the two safety valves (f) would release the excess pressure.

In the meantime, the temperature of the container (a) has dropped and the rate of evaporation is lowered with consequent drop in the pressure in the containers (a), (b), and (c) as the result of continuous consumption.

The moment this pressure approaches the required minimum of 45 lbs., the valve (d) starts to close, raising slightly the pressure in the container (a). This pressure instantly forces the liquid oxygen in the tube (e) spilling liquid into the container (c) and producing a large volume of oxygen gas until enough pressure is produced to open valve (d). If at any time the pressure rises above the critical design pressure, safety valves (f) release the excess pressure. This process is continuous until all the liquid oxygen is used up.

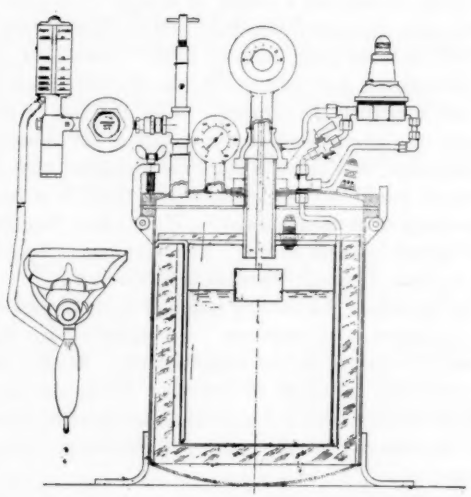

FIG. 3. Schematic working assembly of liquid oxygen supply apparatus connected to B-L-B mask and second stage reducing valve.

Fig. 3 shows the schematic working assembly of the apparatus as shown on Figs. 4 and 5. The additional items are:

1. A perforated metal shield so that cold gas will uniformly surround the container (a).

2. A neck so arranged that liquid oxygen can be poured in from the outside without disturbing the setup.
3. A visual gage to indicate the amount of oxygen in the container. The apparatus can be inclined or inverted.
4. Safety valves.
5. Automatic absolute pressure valve to restrict the gas flow from the container and spill the liquid in the tank.
6. Gas filter.
7. Starting by-pass valve.

Description of Apparatus

Gas Pressure Container

The outside container is made of black iron, 0.065 inch thick, but on actual installation it could be made of steel armor plate, one inch thick, thus utilizing the same material for armor and for the pressure container. The necessary heat conductivity to vaporize the liquid oxygen would not be impaired.

Insulation

The insulation material (glass wool) around the inside liquid reservoir is enclosed in a perforated, galvanized iron container and a low temperature is provided by the cold oxygen gas seeping through this insulation. Such insulation is not as good as a metal vacuum bottle, but, since a certain amount of steady flow of gas is needed, the poorer insulation acts favorably in this installation. Substitution of a metal vacuum bottle for the glass wool insulation is permissible and would be even better if a very small oxygen supply is needed. A vacuum bottle insulation inside of this apparatus would reduce the evaporation, and consequently the gas supply, to an approximate minimum value of 5 per cent per 24 hours as against approximately 10 per cent per hour for glass wool on small installations. A vacuum bottle would help if minimum oxygen volume is of major importance, but it also could be used to produce larger amounts of oxygen the same as with glass wool insulation. It is very questionable if, for all installations, a vacuum bottle would be justified if we keep in mind that such vacuum bottles would have to be designed with no carbon absorption pads and that they must withstand pressure from both sides on the vacuum space.

Liquid Oxygen Container

The liquid oxygen container can be made out of galvanized steel but never of glass because of the risk of cracking during filling. This container is provided with a low-pressure relief valve in order not to allow a big difference of pressure in the inside of the container which would cause excessive spilling of oxygen and wastage of gas through the safety valve. The liquid container and glass wool insulation could be replaced by a vacuum bottle in one unit.

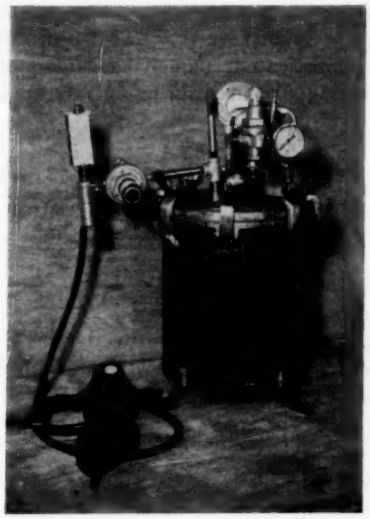

Fig. 4. Experimental apparatus connected to B-L-B mask.

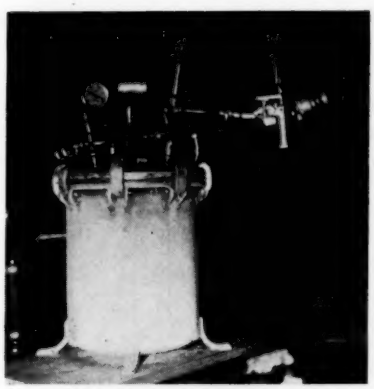

Fig. 5. Experimental apparatus showing slight frost after evaporation of 6 liters of oxygen.

Container Pressure Control Valve

The container pressure control valve should be so constructed that it will open and close the gas outlet from the liquid container in order to spill liquid oxygen through the liquid outlet into the outside container very delicately, to maintain the pressure in the tank within the predetermined limits. Those limits might be set at will from one atmosphere to any desired pressure. This pressure is determined by the requirements of the second-stage reducing valve, passing gas to the mask. With further development of this apparatus there is a possibility of eliminating the second-stage reducing valve and substituting in its place a plain needle valve. This automatic inside pressure control valve is a very essential part of the apparatus and should provide a

smooth supply of gas and avoid excessive waste of oxygen through the safety valves. Two types of valves have been investigated, namely, a sylphon-operated sleeve valve (Fig. 6) and a spring-operated diaphragm

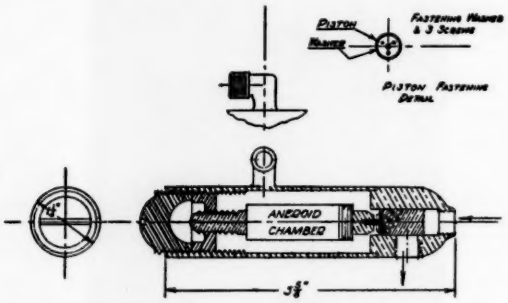

FIG. 6. Plunger valve. Absolute pressure control in oxygen container.

valve. Both have advantages and disadvantages and detailed improvements in either one might make them of equal value.

Containers for Liquid Oxygen Storage

Containers for liquid oxygen storage are on the market in sizes from 5 liters to 100 liters with a rate of evaporation as low as 2 per cent in 24 hours (Figs. 7 and 8).

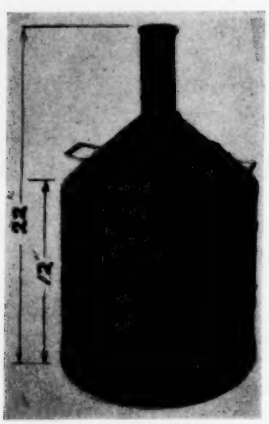

FIG. 7. Small storage container.

Handling of such liquid containers is not any more burdensome in supply bases than handling of gas cylinders since the weight of the liquid container is only a small per cent of the weight of the oxygen, while in gas cylinders the weight of the cylinders is a major part compared with the weight of the actual gas. It is of interest to note that the supply of oxygen gas in cylinders is always secured from a base where first liquid oxygen is delivered and then the gas cylinders are filled with oxygen gas by a process of evaporation and compression. In large scale operations using gas cylinders, liquid oxygen handling and supplying is unavoidable.

Liquid Oxygen Producing Plants

As stated above, the supply of cylinders of gaseous oxygen requires bases where liquid oxygen can be produced or to which it can be shipped from central producing plants in special oxygen tank cars or in large size containers and then evaporated, compressed and filled in oxygen gas cylinders.

FIG. 8.

There is a belief that the supplying of liquid oxygen would be burdensome and almost an impossibility for such places as Army or Navy bases. Such bases would have to be supplied with a filling plant to which liquid oxygen would have to be delivered, and provision would have to be made to convert liquid into compressed gas in cylinders. To the surprise of the author, manufacturers of oxygen equipment, on inquiry, submitted data showing that an oxygen plant, portable on a truck, could be easily manufactured. For example, a plant producing 5 liters of liquid oxygen per hour would be portable on one truck and could be run with the same power plant as the truck itself or with an independent motor.

Safety

There have been a sufficient number of tests made which show that incendiary bullets shot through a liquid oxygen container, if the container is not manufactured with carbon absorption pads, will not present a fire hazard. As far as burning of surrounding objects is concerned, it makes no difference whether the oxygen gas comes from liquid or compressed oxygen gas.

Effect of Compass Location in Aircraft on Dynamic Errors

WALTER WRIGLEY

Sperry Gyroscope Company, Inc.

SUMMARY

A compass may be located, either for convenience or necessity, at some distance from the center of gravity of large modern aircraft. Such a location may give rise to dynamic errors in the readings of the compass due to the accelerations of roll, pitch and yaw of the airplane about its center of gravity. The acceleration of yaw is probably the most serious factor in producing such dynamic errors regardless of whether the compass is located in the nose, the tail or in a wing.

The magnitude of the dynamic errors in compass readings increases in an approximately linear fashion as the distance of the compass from the center of gravity of the airplane is increased. For any particular location of the compass there will be a certain heading of the airplane for which the compass reading will have a maximum error. The dynamic errors may exist as magnitude effects of up to several degrees, or as phase effects of up to almost one hundred eighty degrees.

INTRODUCTION

THE MAGNETIC compass, whose duty it is to furnish indications of azimuth to the pilot, may be located at some distance from the center of gravity of large modern aircraft. If, for reasons of convenience, it is located on the instrument panel in the pilot's cockpit, the compass will generally be well forward of the airplane's center of gravity. On the other hand, it may be desirable to mount the compass as a remote reading instrument near the tail of the airplane or out in one of the wings in order to remove the effects of deviation due to the presence of magnetic materials or stray fields near the cockpit. Even if it were desired to mount the compass at the center of gravity of the airplane, such a location would be highly impractical since the center of gravity might be inaccessibly situated inside the structure or in a space required for other purposes. In small craft the compass would probably never be located more than a few feet from the center of gravity, but in large ships it is more than likely to be mounted sufficiently far from the center of gravity to cause its location to have a marked effect on its performance.

A magnetic compass indicates azimuth by showing the direction of the horizontal component of the earth's magnetic field. Actually, the compass indicates on the compass card the direction of the *projection* of the earth's magnetic field and for this reason the card must be pendulous. Except for dynamic effects the card is perpendicular to the direction of the resultant acceleration acting on it and is, therefore, horizontal under the influence of gravity alone. Practical compasses now in use generally have such short natural periods of oscillation about a horizontal axis in relation to the time required for the resultant acceleration of the instrument to change in direction that the compass card remains perpendicular to the resultant acceleration even for dynamic effects.[1] The card then tilts under the influence of horizontal acceleration components. A tilt component of the compass card about the magnetic north-south axis, due to an east-west acceleration, introduces a projection of the vertical component of the earth's magnetic field in the plane of the card.

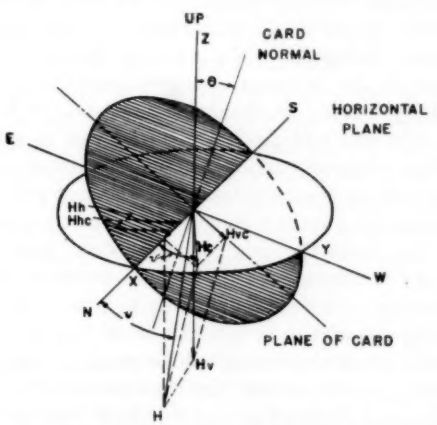

FIG. 1. Projection of earth's field in plane of card. Airplane banked 30° and headed north.[1]

Fig. 1 shows how the projection H_{vc} in the plane of the tilted compass card, due to the vertical component H_v of the earth's magnetic field H, combines with the horizontal component H_h of the earth's field to form a resultant field H_c in the card. Starling[2] and Draper, Cook and McKay[1] have shown that the change in direction from the magnetic north of this resultant field in the compass card is the cause of dynamic errors in compass indications.

ACCELERATIONS ACTING ON A COMPASS CARD

When a compass is located at the center of gravity of an airplane, the instrument experiences only the accelerations of the airplane itself while in flight. Such accelerations accompany changes in linear velocity or

Presented at the Radio and Instruments session, Ninth Annual Meeting, I.Ae.S., New York, January 29, 1941.

changes in the direction of flight during turns. When the compass is located *away* from the center of gravity of the airplane, the instrument will experience, in addition to the above-mentioned accelerations, accelerations due to motions of the airplane about its center of gravity such as rolling, pitching or yawing. Fig. 2 illustrates the origin of these accelerations experienced

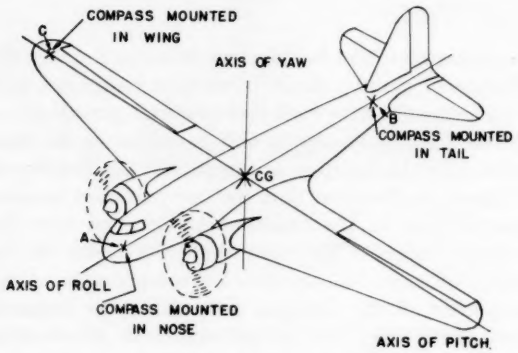

FIG. 2. Various locations of a compass in an airplane.

by a compass when mounted fore, aft or to one side of the center of gravity of the airplane. When mounted in position *A* forward of the center of gravity, or aft in position *B*, the compass is subject mainly to effects caused by yawing of the airplane about its vertical axis. The effect of pitching will add to or subtract from that of gravity and so will be of secondary importance in tilting the compass. The effect of rolling will be small as long as the compass is not located more than a few feet from the axis of roll. When mounted in position *C* to one side of the airplane's center of gravity, the compass is also subject mainly to yaw. The effect of rolling will add to or subtract from that of gravity, while the effect of pitching will be small as long as the compass is not located far from the axis of pitch.

Horizontal acceleration due to yaw is thus seen to be the most important factor in tilting a compass that is located away from the center of gravity of an airplane, although rolling and pitching may also be considered contributing factors. Horizontal acceleration due to yaw is a tangential linear acceleration proportional to the product of the angular acceleration of yaw (a function of the motion of an airplane) and the distance the compass is located from the center of gravity. It is the latter factor that is of interest in this paper.

The tilt of a compass card is dependent on the resultant of *all* the acceleration components—both the horizontal components which depend on the location of the compass in the airplane and the vertical components which are due to the force of gravity.

Compass Performance during a Turn

It has been shown elsewhere by the present writer[3] that from a practical point of view the only horizontal acceleration component effective in tilting a compass card during a turn is the centripetal acceleration of turning; the effects of accelerations due to yaw and roll are negligible. Since the centripetal acceleration acts at the center of gravity of an airplane, it follows that during a turn about the vertical the dynamic errors of a magnetic compass are substantially unaffected by the location of the compass in the airplane.

Compass Performance in an Oscillation about an Axis through the Center of Gravity of an Airplane

To illustrate the effect of an oscillation on the dynamic errors of a compass a sinusoidal yawing of the airplane about its vertical axis is chosen. This yawing motion produces a sinusoidally varying horizontal acceleration component which in turn causes a sinusoidal variation, about a horizontal axis, of the direction of the resultant acceleration acting on the compass. The compass card then executes a sinusoidal tilt about the same horizontal axis, and the north-south component of the tilt introduces a new magnetic field component in the plane of the compass card, tending to cause an error in the compass indications. The magnitude of the dynamic errors of the compass will be linearly proportional to the distance of the instrument from the center of gravity. This is one effect of compass location on dynamic errors. A second effect depends on the heading of the airplane. If the airplane in Fig. 2 is flying in a general northerly or southerly direction, a compass located at points *A* or *B*, *i.e.*, fore or aft of the airplane's center of gravity, will tilt about the north-south axis and so have dynamic errors due to location. However, if the compass is located in a wing at point *C*, the instrument will tilt about an east-west axis which does not lead to dynamic errors. But if the airplane is flying east or west, a compass located at point *C will* have dynamic errors, while one located at points *A* or *B* will not. For intermediate headings all compass locations will experience some north-south tilt components and hence lead to some dynamic errors.

Fig. 3 shows the effect of location on the direction of the resultant magnetic field in the plane of the compass card during a sinusoidal oscillation of the airplane. In Fig. 3 the direction of the resultant field is plotted against the time for one period of yaw. It is assumed that the period of yaw is four seconds, the amplitude of yaw is ten degrees, and the airplane is flying north over New York. It is seen in the figure that the resultant field points north when the compass is located at the center of gravity of the airplane, but the field oscillates about north in a period equal to the period of yaw for non-center-of-gravity locations. For the particular case cited the maximum variation in field direction is about ten degrees when the compass is located

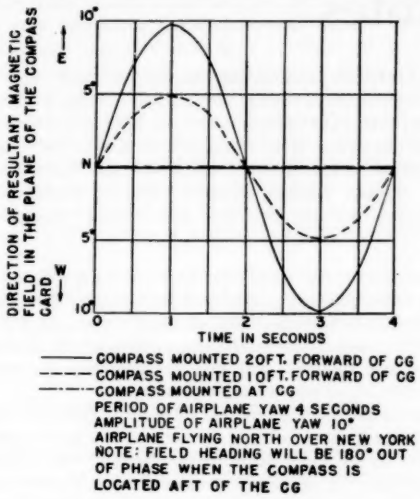
DIRECTION OF RESULTANT MAGNETIC FIELD IN THE PLANE OF THE COMPASS CARD
10°
5°
N
5°
10°
W
0
1
2
3
4
TIME IN SECONDS
COMPASS MOUNTED 20 FT. FORWARD OF CG
COMPASS MOUNTED 10 FT. FORWARD OF CG
COMPASS MOUNTED AT CG
PERIOD OF AIRPLANE YAW 4 SECONDS
AMPLITUDE OF AIRPLANE YAW 10°
AIRPLANE FLYING NORTH OVER NEW YORK
NOTE: FIELD HEADING WILL BE 180° OUT OF PHASE WHEN THE COMPASS IS LOCATED AFT OF THE CG

FIG. 3. Direction of resultant magnetic field in the plane of the compass card as a function of the location of the compass in the airplane during a sinusoidal oscillation in yaw.

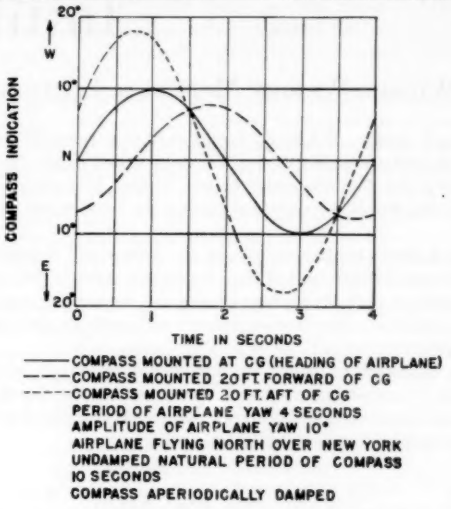

FIG. 4. Dynamic errors in compass indications as a function of location of the compass in an airplane during a sinusoidal oscillation in yaw.

ten feet forward of the center of gravity and about nineteen degrees when located twenty feet forward.

The response of a compass to this field oscillation is shown in Fig. 4. The reading of the compass is plotted against the time for one period of yaw. The compass used is an average of several standard instruments and has an undamped natural period about its axis of symmetry of ten seconds;[1] and, in addition, is critically damped (aperiodic). As seen in Fig. 4, the compass gives correct indications of the heading of the airplane when the instrument is located at the center of gravity of the airplane. When located away from the center of gravity, the compass readings are in error in both magnitude and phase. For the particular case chosen the compass readings show about seventy-five per cent of the true magnitude, but are one hundred thirty-five degrees out in phase when the compass is located forward of the center of gravity. The readings are about double in magnitude, but only forty-five degrees out in phase when the compass is located abaft the center of gravity. When mounted in the wing, the compass gives correct indications. It should be noted here that the curves of Fig. 4 are based on the assumption of steady state oscillations.

If the airplane were flying south under the conditions given in Figs. 3 and 4, the effect of compass location would be reversed. In other words, the compass indications are about double in magnitude and almost correct in phase when the instrument is mounted in the nose. A compass located in the right wing will perform for a westerly course as it does for the northerly course as illustrated in Fig. 4 when mounted in the nose.

REFERENCES

[1] Draper, C. S., Cook, W. H., and McKay, W., *Northerly Turning Error of the Magnetic Compass for Aircraft.* Journal of the Aeronautical Sciences, Vol. 5, No. 9, page 345, July, 1938.

[2] Starling, S. G., *The Equilibrium of the Magnetic Compass in Aeroplanes,* Philosophical Magazine, Vol. 32, page 461, November, 1916.

[3] Wrigley, W., *An Investigation of Methods Available for Indicating the Direction of the Vertical From Moving Bases,* Massachusetts Institute of Technology, Doctor's Thesis, 1941.

Institute Notes

Wilbur Wright Memorial Lecture

Juan T. Trippe, M.I.Ae.S., President of Pan American Airways System, and a member of the Advisory Board of the Institute, delivered the Twenty-Ninth Wilbur Wright Memorial Lecture before the Royal Aeronautical Society in London on June 17, 1941.

Mr. Trippe's lecture on "Ocean Air Transport" described the history and development of Pan American's transoceanic service, the technical methods used in planning and controlling over-ocean flights, and the research on equipment and methods necessary for their improvement.

The invitation to deliver the annual Wilbur Wright Memorial Lecture is one of highest honors conferred by the Royal Aeronautical Society. The lecturers chosen each year are alternately British and foreign aeronautical specialists.

Award for Fog Study

The paper "Fog Study of Miami, Florida," by J. R. DeHart, published in this issue of the Journal, is a report on the research which won for its author an award from the Air Transport Association of America.

Coincident with the reading of his paper at the meeting of the American Meteorological Society in Miami on May 8, it was announced that Mr. DeHart had been selected, for this study, as the first recipient of this annual award with a cash prize of $500. It is conferred by the Air Transport Association to employees of airlines in the United States "for the best original research having practical application to meteorology and dispatching." Recipients are selected by the Operations Committee of the Association, the members of which are the vice-presidents in charge of operations of the nation's airlines.

Mr. DeHart, a Technical Member of the Institute, is a meteorologist on the staff of Eastern Air Lines stationed at Miami, Florida. His paper was part of a general research program started by Eastern Air Lines last summer under the supervision of Joseph J. George, Chief of its Meteorological Department.

DeHart, who is 26, was graduated from the University of Miami with a Bachelor of Science degree in 1937. He then took a year of post-graduate work in electrical engineering at Georgia School of Technology followed by a course in meterology at the Boeing School of Aeronautics. He has been with Eastern Air Lines since the spring of 1939.

New Corporate Members

The "Appreciation Pages" in the front of the Journal now list six additional Corporate Member companies since the last announcement of new Corporate Members made in the March issue.

These companies affiliated with the Institute since that date are:

Associated Aviation Underwriters, New York, N. Y., aeronautical insurance.

Eaton Manufacturing Company, Cleveland, Ohio, manufacturers of aircraft engine parts and accessories, pumps, valves and actuating cylinders.

Casey Jones School of Aeronautics, Newark, N. J., and its affiliated **Academy of Aeronautics,** La Guardia Field, N. Y., which give courses in aeronautical engineering, master mechanics' training, airplane design and construction, airplane mechanics and maintenance.

North American Aviation, Inc., Inglewood, Calif., and Dallas, Texas, constructors of the military aircraft bearing its name.

Packard Motor Car Company, Detroit, Michigan, which is starting the production of Rolls-Royce Merlin XX aircraft engines.

Vultee Aircraft, Inc., Vultee Field, Calif., and Nashville, Tenn., with its **Stinson Aircraft Division,** Wayne, Mich., manufacturing Vultee military airplanes and Stinson commercial airplanes.

A list of the products and services of Corporate Member companies is now published in expanded form in the Directory in the "Aeronautical Review Section" of the Journal. It will be appreciated if Institute members and readers of the Journal and "Aeronautical Review" will give consideration to the items listed in this Directory whenever possible.

Corporate Membership was established in 1936 for companies which wished to participate in the activities of the Institute and contribute to the publication of the Journal. It is limited to "manufacturers or operators of aircraft or producers of equipment, services or supplies used in aeronautics." The membership dues contributed by Corporate Members have enabled the Institute to expand its services over the past years and help make possible the publication of its technical Journal.

The greater contribution to the progress of aeronautics made possible by their support of the Institute is appreciated by all members.

Honorary Degrees to Institute Officers

Major James H. Doolittle, F.I.Ae.S., immediate Past President of the Institute, and Ralph S. Damon, A.F.I.Ae.S., member of the Institute's Council, were the recipients of honorary degrees last month.

Mr. Damon, President of Republic Aviation Corporation and recently Vice-President in charge of Operations for American Airlines, received the honorary degree of Doctor of Engineering from Clarkson College of Technology at its forty-second commencement exercises on June 2, in Potsdam, N. Y. The degree was conferred "for meritorious contributions and outstanding advances in the fields of engineering and science."

The honorary degree of Doctor of Engineering was conferred upon Major Doolittle, who is serving in the Procurement Division of the U.S. Army Air Corps as coordinator of aircraft production in the automotive industry, by the Polytechnic Institute of Brooklyn. At the school's annual Corporation dinner, preceding commencement exercises, Major Doolittle was presented for the degree by Lester D. Gardner, Executive Vice-President of the Institute of the Aeronautical Sciences. In his presentation speech, Major Gardner read letters of congratulations from President Robert G. Sproul of the University of California and President Karl T. Compton of the Massachusetts Institute of Technology, at which schools Major Doolittle had received the degrees of A.B. and D.Sc., respectively; from Harry F. Guggenheim, who recalled the pioneering work in blind landing research done by Major Doolittle for the Guggenheim Foundation; from Alexander Fraser, President of the Shell Oil Company for which Major Doolittle had until recently been Aviation Director; and from Major General H. H. Arnold, as Chief of the U.S. Army Air Corps, in which he referred to Major Doolittle's record in the Air Corps before his resignation in 1931 and expressed pleasure at again having his services available in its present procurement program.

At the commencement exercises on June 11, President H. S. Rogers of Polytechnic Institute of Brooklyn read the following citation:

"Major James Harold Doolittle, speed and safety of flight have focused your energies and challenged your genius. You, above all others, have contributed to their development in commerce and in defense. You have conquered distance and defied time. Your accomplishments have been made, not through good fortune befalling dauntless courage, but through careful use of advanced scientific knowledge and of precise engineering measurements. Your flights and experiments leading up to them have made aviation history and have kept this nation in the forefront of aviation. You have brought to this exacting field of endeavor the highest achievements of intellect and the finest qualities of courageous spirit.

"Acclaiming your pre-eminence in aviation and applauding the manner in which you carry the signal honors you have received, in the name of Polytechnic Institute I confer upon you the degree, Doctor of Engineering, with all its rights and privileges."

News of Institute Members

Cecil W. Armstrong, M.I.Ae.S., Assistant Professor Engineering, University of California, is employed for the summer months as a Senior Research Engineer at the Lockheed Aircraft Corp.

Ensign Dan L. Belvin, U.S.N.R., Technical Member, who was formerly assigned as an Engineering Officer in the Maintenance Division, U.S. Naval Air Station, Pensacola, is now on duty in Washington at the Bureau of Aeronautics, Navy Department.

Major General George H. Brett, who, at the time he was elected an Honorary Member of the Institute, had been serving as Acting Chief of the U.S. Army Air Corps, was appointed on May 23 as Chief of the Air Corps for a regular four year term.

Ray W. Brown, M.I.Ae.S., has left his position as General Manager of the Aviation Department of The General Tire & Rubber Company to go on active duty in the Bureau of Aeronautics, Navy Department.

William A. M. Burden, M.I.Ae.S. and a Vice-President of the Institute, has resigned as a Director of National Aviation Corp. He is connected with the recently established unit of the Reconstruction Finance Corporation concerned with financial aid to Latin American aviation.

John E. Cameron, Technical Member, has left the employ of Boston-Maine Airways to join the engineering staff of Boeing Aircraft Company as a layout man.

Lt. Clinton D. Case, U.S.N., A.F.I.Ae.S., has been detailed to duties aboard the U.S.S. Albemarle. He had previously been Officer-in-Charge, Power Plant Division, Ground School, U.S. Naval Air Station at Pensacola.

Charles Healy Day, A.F.I.Ae.S., has recently been appointed to the Canadian Government's Department of Munitions and Supply as Supervisor of Production, Aircraft Branch, with headquarters in Ottawa. Mr. Day established an aircraft factory for the Chinese Government at Shiuchow in 1934 and for the next six years was technical adviser on aircraft production to the National Government of China.

Dr. J. P. Den Hartog, F.I.Ae.S., Associate Professor of Applied Mechanics, Harvard University, has been assigned to active duty in Washington as a Lieutenant Commander, U.S. Naval Reserve, at the Bureau of Ships, Navy Department.

George A. Doole, Jr., M.I.Ae.S., has been appointed Operations Manager for Aerovias Nacionales de Colombia of the Pan American Airways System at Barranquilla. He was formerly Chief Pilot of AVIANCA.

At the annual meeting of the Aviation Writers Association held in Louisville, Ky., June 1–4, 1941, **Michael Froelich,** M.I.Ae.S., News Editor of Aero Digest was elected Treasurer and **Carl Norcross,** M.I.Ae.S., Managing Editor of Aviation, Secretary of the Association.

L. W. Glowa, Technical Member, formerly with the Sales Department of American Airlines in New York, is now an Airways Engineer for the U.S. Army Air Corps in Washington.

A. I. Lodwick, M.I.Ae.S., formerly with Aviation Manufacturing Corporation, has recently established the Lakeland School of Aeronautics at Lakeland, Florida, in partnership with H. S. Darr.

R. C. McKenzie, Technical Member, formerly with De Havilland Aircraft of Canada Ltd. is now serving as an aeronautical engineer with the Royal Canadian Air Force at Montreal.

Oliver L. Parks, M.I.Ae.S., President of Parks Air College, was recently elected President of the Southwest Aviation Conference. **L. G. Fritz,** M.I.Ae.S., Vice-President in charge of Operations for TWA, was elected Vice-President of the Conference.

E. Robert Reff, M.I.Ae.S., formerly Structural Development Engineer with the Airplane Division of the Curtiss-Wright Corporation at Buffalo, has joined the engineering staff of Northrop Aircraft, Inc.

Laurance S. Rockefeller, M.I.Ae.S., has been elected a member of the Executive Committee of the Board of Directors of McDonnell Aircraft Corp.

A. P. Taliaferro, Jr., M.I.Ae.S., has resigned from the vice-presidency of Aeronautical Securities, Inc., to go on active service in the U.S. Navy as Lieutenant Commander.

At a recent meeting of the Board of Directors of Wright Aeronautical Corp., **Philip B. Taylor,** F.I.Ae.S., Assistant General Manager, was elected Vice-President, and **William D. Kennedy,** M.I.Ae.S., Sales and Service Manager, was elected Vice-President in charge of Sales.

Herbert V. Thaden, A.F.I.Ae.S., formerly Assistant Sales Manager of Hayes Manufacturing Corp., has been appointed Manager of the Commercial Department of Duramold Aircraft Corp., New York.

Charles F. Thomas, M.I.Ae.S., Technical Sales Engineer with Lockheed Aircraft Corp., is now in charge of the company's office in Dayton.

Lt. Comdr. D. W. Tomlinson, M.I.Ae.S., has been appointed Commanding Officer of the U.S. Naval Reserve Aviation Base at Fairfax Airport, Kansas City. He is on indefinite leave of absence from his office as Vice-President in charge of Engineering for TWA, Inc.

Howard M. Willson, Technical Member, has been transferred from the Cleveland Airport office of American Airlines to the company's office at Lockheed Air Terminal, Burbank, California.

Honorary Members Elected

Robert A. Lovett, Assistant Secretary of War for Air, and Merrill C. Meigs, Chief of the Aircraft Section, Office of Production Management have been elected to Honorary Membership in the Institute.

Merrill Church Meigs was born in Malcolm, Iowa, November 25, 1883. He studied at Iowa Business College and in business courses at the University of Chicago. For the past thirty years he has been prominent in advertising and publishing activities in Chicago. He was successively publisher of the Chicago Herald and Examiner from 1926 to 1929; of the American Weekly from 1930 to 1933 and of the Chicago American from 1933. Since his first airplane flight in 1928 he has been an enthusiastic amateur pilot, having obtained a Transport Pilot's license and accumulated close to 1300 hours of solo flying time. He has been chairman of the Chicago Aeronautics Commission and Illinois Aeronautics Commission. He was named to head the Aircraft Section of OPM in January of this year.

Robert Abercrombie Lovett was born September 14, 1895, at Huntsville, Texas, and received a B.S. degree from Yale University in 1918 followed by studies in law and business administration at Harvard University. During the World War he was a pilot in the U.S. Naval Air Service, rose to the rank of Lieutenant Commander and was awarded the Navy Cross for his service overseas. His extensive experience in finance and industry began in 1921 with a clerkship in the National Bank of

Commerce in New York and has included membership on the boards of directors of several railroads and manufacturing companies, insurance and banking institutions. He has also been active in numerous charitable organizations in New York City.

Mr. Lovett was a partner in the firm of Brown Brothers, Harriman & Co. until called in December, 1940, to serve as a special assistant to Secretary of War Stimson. His accomplishments in this position in handling executive matters pertaining to the expansion of the Air Corps led to his appointment by President Roosevelt on April 10, 1941, to the long vacant post of Assistant Secretary of War for Air. In this office his efforts will be of great importance in the unification of authority within the Air Corps and better coordination with the entire national defense effort.

I.Ae.S. Student Branch Awards

Recipients of the two Student Branch awards, the I.Ae.S. Student Branch Scholastic Award and the I.Ae.S. Student Branch Lecture Award, in addition to those listed in the June issue of the Journal, are as follows:

Student Branch Scholastic Award: **B. Shreve Hutchins,** Agricultural and Mechanical College of Texas; **Frank T. Lewand,** University of Detroit; **Robert G. Urquhart,** Harvard University; **Leon R. Wosika,** University of Michigan; **Robert W. Bratt,** University of Minnesota; **Walter Walling,** Oregon State College; **LeRoy Krouse,** Tri-State College; and **Victor M. Ganzer,** University of Washington.

Student Branch Lecture Award: **Jack Holt,** Agricultural and Mechanical College of Texas; **William S. Holmes,** Polytechnic Institute of Brooklyn; **Frank T. Lewand** and **Frank J. Bajkowski** (jointly), University of Detroit; **J. Norman Ball, Jr.,** Harvard University; **Leslie J. Trigg,** University of Michigan; **James A. Butcher,** University of Minnesota; **Raymond Pederson,** Oregon State College; **Warren LeRoy,** Tri-State College; and **William F. Kaseburg,** University of Washington.

Personnel Opportunities

The Personnel Bureau serves individual members, as well as organizations seeking to employ aeronautical specialists. Any member or organization may have requirements listed without charge.

Wanted

Foreman—Balloons—completely familiar with construction of lighter-than-air craft. Location in East. State experience and salary desired. Address reply to Box 135, Institute of the Aeronautical Sciences.

Student Branches

Boeing School of Aeronautics. At a recent meeting, the following new officers were elected: Andrew Smisko, Chairman; Charles Strausser, Vice-Chairman; and Claude Williams, Secretary-Treasurer.

University of California. Charles W. Harper, Chairman, appointed Bradford Wick and Arthur Hughes as a committee of two to organize the Student Branch for the Fall semester.

University of Detroit. George H. Tweney, Instructor in Aeronautics, presented the I.Ae.S. Branch Lecture on "Stainless Steel and its Aircraft Uses" by Dr. Michael Watter, at a recent meeting. New officers were elected as follows: James Hailstones, Chairman; James Rice, Vice-Chairman; Wilfred Bush, Treasurer; and Louis B. Peltier, Secretary. Prof. George J. Higgins was re-elected Faculty Sponsor.

University of Michigan. The following men have been elected to serve as officers for the school year 1941–42: Henry T. Fielding, Chairman; J. Schwarzbach, Treasurer; and Arthur Billet, Secretary. Elections for the office of Vice-Chairman and Honorary Chairman will take place in the Fall.

New York University. At a meeting held on May 7, officers were elected as follows: James Smith, Chairman; Ralph Nunziato, Vice-Chairman; and Frank Shieflet, Secretary-Treasurer.

Oregon State College. On May 12, officers were elected as follows: Lawrence Robertson, Chairman; Forbes McRae, Vice-Chairman; and Don Jacoby, Secretary-Treasurer. Prof. Benjamin F. Ruffner was re-elected Honorary Chairman.

Ryan School of Aeronautics. A dinner meeting was held on April 18 at which time Walter B. Sayner read a paper on "The Assisted Take-Off." At a later meeting, held on May 23, W. T. Immenschuch gave a paper on "The Work of the Designing Engineer."

Virginia Polytechnic Institute. Officers for the next school year were elected as follows: J. W. Martin, Jr., Chairman; E. F. Burner, Vice-Chairman; A. Hertzberg, Secretary; and W. W. Lumpkins, Treasurer.

Changes in Membership

The following changes in the membership of the Institute have occurred since the publication of the previous issue of the Journal.

Elected to MEMBER Grade

Dufort, Robert Harold, B.S. in Ae.E.; Head of Stress Dept., Goodyear Aircraft Corp.

Markey, Richard Joseph, Final Assembly Production Planning, Vought-Sikorsky Aircraft, United Aircraft Corp.

Montgomery, Wiley Pope, B.S. in Ae.E.; In charge of Structural Research, Vought-Sikorsky Aircraft, United Aircraft Corp.

Shields, Bert A., Aviation Writer; Ground Flying Specialist and Lecturer, Columbia University; New York University.

Stout, Ernest Gordon, Ae.E.; Engineer in charge of Hydrodynamics, Consolidated Aircraft Corp.

Elected to Industrial Member Grade

Favorite, Lewis P., Sales (Engineering Service), Aluminum Co. of America.

Warner, Percy deForest, II, Asst. Chief, Ground School Sec., Standards Div., Civilian Pilot Training, C.A.A.

Elected to Technical Member Grade

Allen, Harold Frederick, M.S. in E.E.; Instructor, University of Michigan.

Brown, Chester Harvey, Jr., B.S.; Junior Aero. Engineer, Grumman Aircraft Engineering Corp.

Carder, Alden Bivins, B.S. in Ae.E.; Flight Test Engineer, Douglas Aircraft Co.

Delany, Noel Keesling, B.S. in Ae.E.; Jr. Aero. Engineer, Ames Aero. Lab., N.A.C.A.

Dunn, Louis Gerhardus, Ph.D.; Instructor in Aeronautics, Calif. Inst. of Technology.

Fruauff, George Louis, Jr., Bench Metal Man, Brewster Aeronautical Corp.

Graham, William Henry, M.A.; Assoc. Editor, Aeronautical Review, Institute of the Aeronautical Sciences.

Hook, Franklin Louis, B.S. in Ae.E.; Draftsman, The Glenn L. Martin Co.

Johnson, Paul Alford, B.S.E.; Vibration & Stress Engineer, Boeing Aircraft Co.

Katzenberger, Edward Frederick, B.M.E.; Senior Layout Engineer, Brewster Aeronautical Corp.

Kruchek, Keith Cornelius, B.S. in Ae.E.; Engineer (Structures), Boeing Aircraft Co.

Lindblom, Charles William, S.B. in Ae.E.; Shop Liaison Engineer, Lockheed Aircraft Corp.

Markowitz, Jesse, M.S. in M.E.; Instructor, School of Technology, City College of New York.

Myers, Oscar Bruce, Jr.; B.S. in M.E.; Military Pilot, U.S. Army Air Corps.

Nicol, William Franklin, Chief Engineer, Braniff Airways, Inc.

Nietsch, Herman Erich, S.B.; Aero. Engineer, The Glenn L. Martin Co.

Paradiso, Michael Antonio, Layout Design Engineer, Vought-Sikorsky Aircraft, United Aircraft Corp.

Roberts, Phillip H., B.S.E.; Senior Detail Draftsman, Lockheed Aircraft Corp.

Rohl, Henry Thomas, B.S. in Ae.E.; Jr. Aero. Engineer, U.S. Army Air Corps, Materiel Div., Wright Field.

Sawyer, James Garnett, B.S. in M.E.; Airplane Division (St. Louis Plant), Curtiss-Wright Corp.

Smith, George Campbell, B.S. in Ae.E.; Engineer Officer, Royal Canadian Air Force. Canada.

Street, Robert Elliott, Ph.D.; Instructor in Math., Rensselaer Polytechnic Inst.

Werner, Louis Fred, President & Gen. Manager, Models, Inc.

Williams, Richard Samuel, Ae.E.; Technical Data Engineer, Intercontinent Aircraft Corp.

Wright, Frederick Hamilton, Aerodynamics Dept., Douglas Aircraft Co.

Necrology

Herbert Marsh

Herbert Marsh, a member of the Institute, Superintendent of Instruction at the Boeing School of Aeronautics, died on June 10, 1941, at his home in San Leandro, California.

Mr. Marsh was born in Lancashire, England, December 22, 1886, and educated in the Viduct School of Technology there. At the outbreak of the World War in 1914 he left the Daimler Motor Company of Coventry, England, to become Chief Examiner in the Aeronautical Inspection Department of the British Air Ministry. With a commission as Captain, he was Chief Technical Officer in the Materiel Division (Research Supply) of the Royal Flying Corps and the Royal Air Force from 1916 to 1919. Following this he was a Works Inspector for the Air Ministry.

On coming to the United States, he joined the National Parks Airways in 1929 as Assistant Manager of Operations. He became an Instructor on the staff of the Boeing School of Aeronautics in 1935 and had been with the school from then until the time of his death.

Book Review

Theodor von Kármán Anniversary Volume; Contributions to Applied Mechanics and Related Subjects, by the Friends of Theodore von Kármán on his Sixtieth Birthday; California Institute of Technology, Pasadena, 1941; 337 pages, $3.75.

The twenty-six scientific papers presented in this volume are preceded by a short biographical note by Clark B. Millikan with a list of ninety-one important publications by von Kármán in the fields of Applied Mathematics, Physics, Strength of Materials, Stress Analysis, Theory of Elasticity, Elastic Stability, Monocoque Structures, Vibrations, Mechanics of Ideal, Viscous, and Compressible Fluids, Turbulence, Aerodynamics of Aircraft, Hydrodynamics of Planing Surfaces, and Heat Transfer.

The Table of Contents of this volume best indicates its character:

H. Bateman, California Institute of Technology:
Some Definite Integrals Occurring in Aerodynamics

Max M. Munk, Catholic University of America:
On the Geometry of Streamlining

J. C. Hunsaker, Massachusetts Institute of Technology:
Dimensional Analysis and Similitude in Mechanics

Theodore Theodorsen, National Advisory Committee for Aeronautics:
Impulse and Momentum in an Infinite Fluid

Boris A. Bakhmeteff, Columbia University:
Coriolis and the Energy Principle in Hydraulics

H. U. Sverdrup, Scripps Institution of Oceanography of the University of California:
The Influence of Bottom Topography on Ocean Currents

W. F. Durand, Stanford University:
Ionization as a Factor in Fluid Mechanics

Hugh L. Dryden, National Bureau of Standards:
Isotropic Turbulence in Theory and Experiment

J. L. Synge and W. Z. Chien, University of Toronto:
The Intrinsic Theory of Elastic Shells and Plates

Francis D. Murnaghan, Johns Hopkins University:
The Compressibility of Solids under Extreme Pressures

F. Zwicky, California Institute of Technology:
Hydrodynamics and the Structure of Stellar Systems

H. M. Westergaard, Harvard University:
On the Elastic Distortion of a Cylindrical Hole by a Localized Hydrostatic Pressure

Paul S. Epstein, California Institute of Technology:
On the Absorption of Sound Waves in Suspensions and Emulsions

R. Courant, New York University:
On a Method for the Solution of Boundary-Value Problems

Karl Arnstein, Goodyear Aircraft Corporation:
The Engineering Treatment of Ring or Wheel Problems

A. Einstein, V. Bargmann, and P. G. Bergmann, The Institute for Advanced Study:
On the Five-Dimensional Representation of Gravitation and Electricity

S. Timoshenko, Stanford University:
The Forced Vibrations of Tie-Rods

Th. Troller, Daniel Guggenheim Airship Institute:
A Note on the Effect of the Wind-Tunnel Size on Pitching Moments

A. Nadai, Westinghouse Research Laboratories:
The Creep of Metals under Various Stress Conditions

K. O. Friedrichs, New York University:
On the Minimum Buckling Load for Spherical Shells

William Bollay, Harvard University:
The Theory of Flow through Centrifugal Pumps

Frank L. Wattendorf, Wright Field:
The Ideal Performance of Curved-Lattice Fans

L. H. Donnell, Illinois Institute of Technology:
Stress Concentrations Due to Elliptical Discontinuities in Plates under Edge Forces

Hans Reissner, Illinois Institute of Technology:
On Lubrication Flow with Periodic Distribution between Prescribed Boundaries

R. v. Mises, Harvard University:
Some Remarks on the Laws of Turbulent Motion in Tubes

Wolfgang B. Klemperer, Douglas Aircraft Company:
Stress Pattern Crazing

J. C. Hunsaker

SUGGESTIONS FOR CONTRIBUTORS TO THE JOURNAL *of the* AERONAUTICAL SCIENCES

The Institute of the Aeronautical Sciences invites both members and non-members from any country to submit papers for publication in the Journal of the Aeronautical Sciences. The Journal, following the practice of other scientific publications, does not pay for contributions.

The following directions for the preparation of papers, if followed by authors, will save correspondence, avoid the return of papers for changes, minimize the work of preparation for the printer, and save the expense due to the charges made for "author's corrections."

MANUSCRIPTS: Papers must be written in English, in original typewriting on one side only of white paper sheets, consecutively numbered, and be double or triple spaced with wide margins. Manuscripts should be prepared with great care so that they will be typographically accurate. Paragraphing should be given special attention. Papers should be written in the third person, reference to the writer being made as "the author." Avoid the use of the words "I," "we," and "you." Blueprint copies of papers are unacceptable as it is impossible to mark directions to the printer on them. Correcting, changing, or adding to papers after they are in type is costly. It is, therefore, imperative that papers submitted be in final form. Typographical errors may be corrected on proofs, but if authors wish to add material, they may do so at their own expense. In mailing, drawings may be rolled, but manuscripts should be sent flat. Send by first class mail (register if you wish for your own protection) to the Secretary, Institute of the Aeronautical Sciences, 1505 RCA Bldg. West, Rockefeller Center, New York City. All manuscripts will be examined by the Editorial Committee and by the Editor. Authors will be advised as promptly as possible (usually two to three weeks) whether the paper is acceptable for publication.

TITLES: The title of the paper should be brief. The name and initials of the author should be written as he prefers. The use of the full name of an author is advocated because of the frequent duplication of initials and surnames which sometimes makes it difficult to establish the identity of the author. This is particularly important for large annual indexing and abstracting services. All titles and degrees or honors are omitted. The name of the organization with which the author is associated should be placed after his name on the same line. The date on which the paper is received will be inserted by the Editor.

SUMMARIES OR ABSTRACTS: An abstract to be printed at the beginning should accompany each article. It should be adequate as an *index* and as a *summary*. It should contain a statement of major conclusions reached, since summaries in many cases constitute the only source of information used in compiling scientific reference indexes. Abstracts printed in other journals, especially foreign, in most cases, consist of summaries from printed papers. The summary should explain as adequately as possible the major conclusions to a non-specialist in the subject. The summary should contain from 100 to 300 words, depending on the length of the paper.

SUB-HEADINGS: Sub-headings should be inserted by the author at frequent intervals. The work of editorial preparation will be simplified by the author providing many sub-headings. Owing to the breaking of columns and the insertion of illustrations, some of the sub-headings may have to be omitted.

SHORTENING OF PAPERS: Some papers, at the end, fill in only a portion of a page. This leaves much wasted blank space as succeeding articles are started at the top of a page. Authors should indicate by notation on the left-hand side of the page what matter may be omitted when "run overs" occur. This request is important as the Journal cannot afford in the future, as it has in its earlier issues, to have blank half pages or more at the end of papers.

MATTER USUALLY DELETED: Acknowledgments of assistance in preparation of paper, except by collaborators. Photographs or illustrations of little technical interest and not showing advances in general practice. Too detailed tabular matter (general results of such tables may be included in the text). Lengthy descriptions of materials or processes or of preliminary experiments or theories which preceded final results; salient features only are of interest.

REFERENCES AND FOOTNOTES: References should appear as footnotes only, numbered consecutively, grouped together at the end of the manuscript. The arrangement should be as follows: (for books)—[1] Durand, W. F., *Aerodynamic Theory*, Vol. 1, p. 23; Julius Springer, Berlin, 1934. (For magazines)—Englund, C. R., Crawford, A. B., and Mumford, W. W., *Some Results of a Study of Ultra-Short-Wave Transmission Phenomenon*, Proc. I. R. E., Vol. 20, No. 12, pp. 481 and 482, March, 1933. Please give Author, Title, Volume, page, publisher and date of publication as indicated. Omission of one required fact causes much extra editorial work and possible inaccuracies. All references are grouped at the end of the article.

ILLUSTRATIONS: Illustrations should accompany manuscripts and each should always be referred to in the text, preferably by number. Drawings or graphs should not be larger than 12 × 16 inches, and must be made with jet black India Ink on white paper or tracing cloth, the latter being preferred. *Do not use typewriter for lettering.* The smallest lettering on 8 × 10 inch figures should be no less than $1/4$ inch high. Cross-section paper (white with black lines) may be used, but should not have more than 4 lines per inch. If finer ruled paper is used, the major division lines should be drawn in with black ink, omitting the finer divisions. In the case of finely ruled paper, only blue-lined paper can be accepted. *Tracing paper and blueprints* are not acceptable. Lettering and all markings must be large enough to be readable after reduction. Mail rolled or flat, never fold. Drawings which cannot be reproduced (including pencil drawings) will be returned to the author for redrawing, thus delaying publication of the paper. Photographs should be very distinct and show clear black and white contrasts. They must be on glossy white paper. Avoid round and oval photographs.

CAPTIONS AND LEGENDS: Legends or captions must accompany each drawing or photograph submitted. If written on the drawing or photograph, they should be placed below and well outside the part to be reproduced. It is better to place them on separate sheets of paper pasted to the back of the drawings or photographs. Each table should have a caption such as Table 1, Table 2, Table 3, etc. Captions should be complete in themselves so as to make the data intelligible to the reader without reference to the text. A duplicate list of captions for figures should be included as the last page of the manuscript. Use "Fig. 1" (not Figure 1), Figs. 3 and 4, etc., in both the text and the numbering of illustrations. In the text, "Eq. (1)," or "Eqs. (1) and (2)" are preferable to "Equation (1)." In captions and legends, except for "Fig." and "Eq." and table headings, write all words in full; do not abbreviate. Avoid placing explanatory written matter in the drawings; it should be in the text.

MATHEMATICAL WORK: Only the very simplest formulae should be typewritten; all others should be very carefully written in pen and ink, the writing to be large enough so that ample room is provided to mark mathematical matter for the printer. A considerable space for marking should be allowed above and below all equations. All complicated equations should be repeated on separate sheets with plenty of space left for marking. The solidus should be used for simple fractions appearing within the text. Make all expressions clear to the typesetter. Greek letters used in formulae should be clearly designated by name on the margin of the manuscript. All symbols should be clearly written and carefully checked. The difference between capital and lower-case letters should be clearly distinguished and care taken to avoid confusion between zero (0) and the letter (o), between the numeral (one) and the letter (ell) and the prime (′), between alpha and a, kappa and k, u and mu, v and nu, n and eta. All subscripts and exponents should be clearly marked and dots and bars over letters or mathematical expressions should be avoided. Avoid complicated exponents and subscripts. When it is necessary to repeat a complicated expression, it should be represented by some convenient symbol.

NOMENCLATURE AND ABBREVIATIONS: The National Advisory Committee for Aeronautics Nomenclature should be used in preference to any others. Standard abbreviations should be used, and it should be noted that most abbreviations are lower case, such as m.p.h., b.m.e.p., i.hp., b.hp., hp., ... etc.